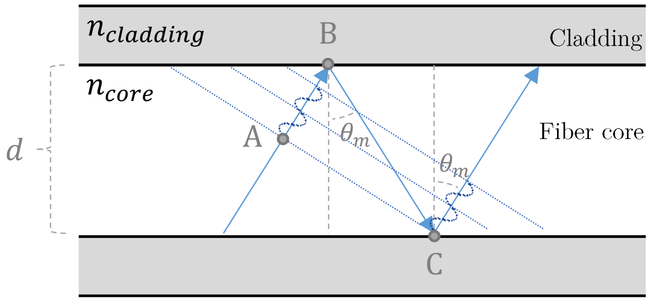
Photonics, Spring 2023

MODEL SOLUTIONS: Exercise 6, 28.2.2023

1. Waveguide condition (2 point)

Two weeks ago (Exercise 4) we looked at optical fiber waveguides. Consider a single ray traveling along a waveguide undergoing total internal reflections at angle θ_m. Evidently for a coherent wave to propagate along the waveguide, the wavefronts from successive total internal reflections must interfere constructively. We have also learnt that TIR imposes a dependent phase-shift on the reflected ray, which is dependent on the incident angle θ_m.



Derive the waveguide condition for a slab waveguide (thickness d, refractive indices n_{core} and $n_{cladding}$, incident angle with respect to the cladding is θ_m, vacuum wavelength λ_0, and the phase change upon reflection is ϕ_m):

$$\frac{2\pi n_{core} d}{\lambda_0}\cos\theta_m - \phi_m = m\pi$$

Hint: Consider the ray propagating along ABC. At point C there are two contributions to a phase shift relative to A; one arising from the optical path difference (OPD), and the other from reflections. How much phase shift would an OPD of λ cause? How about then an arbitrary OPD (ABC)? Note that: $\cos(2x) = 2\cos^2 x - 1$.

Let's first figure out the phase-shift contribution from the OPD. An OPD of λ would correspond to a phase shift of 2π. The phase-shift of an arbitrary OPD is then that distance divided by λ and converted to radians with a factor of 2π:

$$\Delta\phi_{m,OPD} = \frac{2\pi * OPD}{\lambda} = \frac{2\pi * ABC}{\lambda} = \frac{2\pi(AB + BC)}{\lambda} = \frac{2\pi n_{core}(AB + BC)}{\lambda_0}$$

We can express AB and BC as trigonometric functions:

$$\cos\theta_m = \frac{d}{BC} \quad \rightarrow \quad BC = \frac{d}{\cos\theta_m}$$

$$\cos 2\theta_m = \frac{AB}{BC} \quad \rightarrow \quad AB = BC\cos 2\theta_m = \frac{d\cos 2\theta_m}{\cos\theta_m}$$

$$\rightarrow AB + BC = \frac{d}{\cos\theta_m} + \frac{d\cos 2\theta_m}{\cos\theta_m} = \frac{d}{\cos\theta_m}(1 + \cos 2\theta_m) \quad \| \quad \cos(2x) = 2\cos^2 x - 1$$

$$AB + BC = \frac{d}{\cos\theta_m}(2\cos^2\theta_m) = 2d\cos\theta_m$$

$$\rightarrow \Delta\phi_{m,OPD} = \frac{4\pi n_{core} d \cos\theta_m}{\lambda_0}$$

Each reflection introduces a phase shift of $-\phi$ (negative due to phase lag), so comparing points A and C there are two reflections:

$$\Delta\phi_{m,TIR} = -2\phi$$

We then note that for constructive interference to occur, the phase difference between points A and C must be an integer multiple of 2π:

$$\Delta\phi_m = \Delta\phi_{m,OPD} + \Delta\phi_{m,TIR} = 2\pi m$$

$$\frac{4\pi n_{core} d \cos\theta_m}{\lambda_0} - 2\phi = 2\pi m$$

$$\frac{2\pi n_{core} d}{\lambda_0} \cos\theta_m - \phi = \pi m$$

2. Waveguide modes (2 point)

Consider a planar dielectric waveguide with a core thickness of 20 μm, $n_{core} = 1.455$, $n_{cladding} = 1.440$, $\lambda = 900$ nm. Recall that the expression for phase-shift ϕ upon TIR, for a TE mode (s-polarized) is:

$$\tan\left(\frac{\phi_m}{2}\right) = \frac{\sqrt{\sin^2\theta_m - \left(\frac{n_{cladding}}{n_{core}}\right)^2}}{\cos\theta_m}$$

Numerically (graphically) compute the solution for θ_m and plot the results.
Hint: Using the waveguide condition (from Exercise 1), you can express ϕ_m as a function of θ_m and m. The left and right side are then both functions of θ_m. What do the intercepts of these functions correspond to? Note that the left function will be periodic, so solutions of odd and even values for m will overlap. The solutions for θ_i will be between 80° and 90°.

Using the waveguide condition:

$$\frac{2\pi n_{core} d}{\lambda_0}\cos\theta_m - \phi = m\pi$$

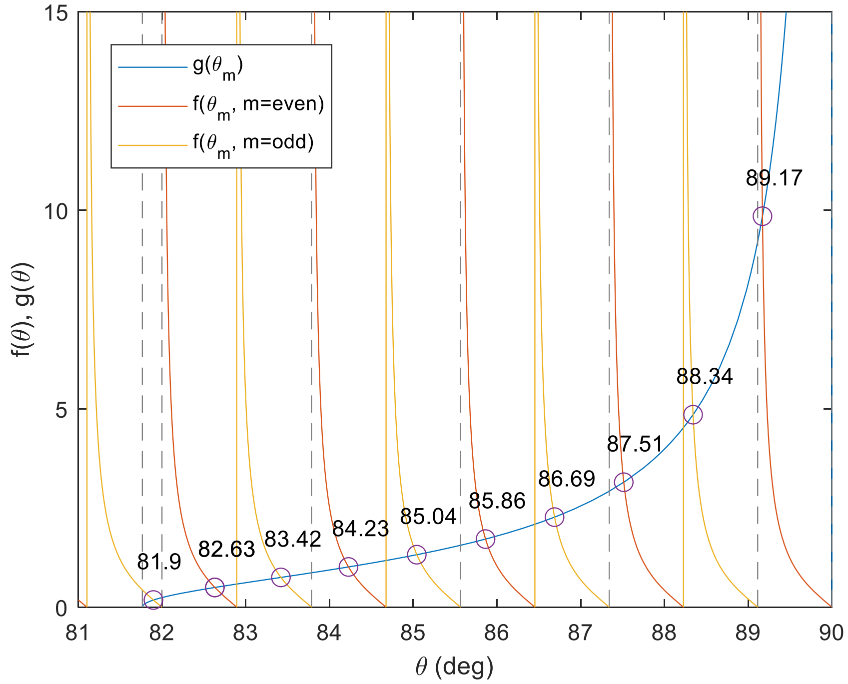
$$\rightarrow \phi = \frac{2\pi n_{core} d}{\lambda_0}\cos\theta_m - m\pi$$

$$\tan\left(\frac{\phi_m}{2}\right) = \tan\left(\frac{\pi n_{core} d}{\lambda_0}\cos\theta_m - \frac{m\pi}{2}\right)$$

We then define our two functions, where their intercepts will give solutions to θ_m:

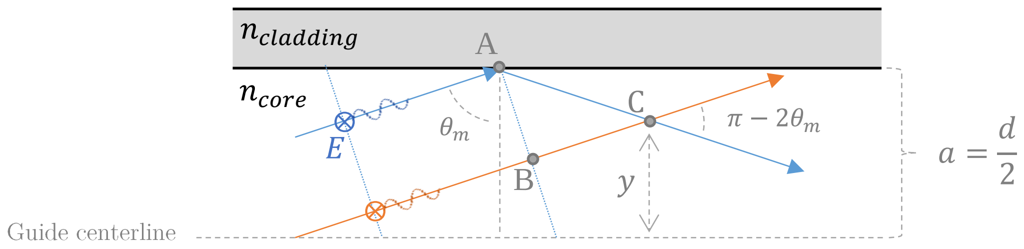
$$f(\theta_m, m) = \tan\left(\frac{\pi n_{core} d}{\lambda_0}\cos\theta_m - \frac{m\pi}{2}\right) \quad , \quad g(\theta_m) = \frac{\sqrt{\sin^2\theta_m - \left(\frac{n_{cladding}}{n_{core}}\right)^2}}{\cos\theta_m}$$

Plotting the two functions:



3. Waveguide mode shapes (4 points)

Let's continue the analysis from Exercise 1 to solve the actual electric field wavefunctions corresponding to different propagating modes. Consider now two identical parallel rays traveling in a waveguide.



a) Show that the phase difference between rays 1 and 2 at C is $\Delta\psi_m = \Delta\psi_m(y) = m\pi - \frac{y}{a}(m\pi + \phi_m)$.
Hint: Use geometrical reasoning to derive ϕ_m as a function of θ_m, then use the waveguide condition to get rid of the explicit θ_m dependence. You'll probably need the identity: $\cos(2x) = 2\cos^2 x - 1$

Again we have two contributions to the phase difference; one from the OPD, and one from the singular reflection. As in Exercise 1, the phase difference arising from the OPD is:

$$\Delta\psi_{m,OPD} = \frac{2\pi n_{core}\mathrm{OPD}}{\lambda_0} = \frac{2\pi n_{core}(\mathrm{AC} - \mathrm{AB})}{\lambda_0}$$

AC is easy enough to figure out:

$$\cos\theta_m = \frac{a-y}{AC}$$

$$\rightarrow AC = \frac{a-y}{\cos\theta_m}$$

BC can then be solved as a function of AC:

$$\cos(\pi - 2\theta_m) = \frac{BC}{AC}$$

$$\rightarrow BC = AC\cos(\pi - 2\theta_m) = \frac{a-y}{\cos\theta_m}\cos(\pi - 2\theta_m)$$

$$= -\frac{a-y}{\cos\theta_m}\cos(2\theta_m) \quad \| \quad \cos(2x) = 2\cos^2 x - 1$$

$$= -\frac{a-y}{\cos\theta_m}(2\cos^2\theta_i - 1) = -\frac{2(a-y)}{\cos\theta_m}\cos^2\theta_i + \frac{a-y}{\cos\theta_m}$$

$$BC = \frac{a-y}{\cos\theta_m} - 2(a-y)\cos\theta_m$$

Now the path difference between AC and BC will give the OPD:

$$\rightarrow AC - BC = \frac{a-y}{\cos\theta_m} - \frac{a-y}{\cos\theta_m} + 2(a-y)\cos\theta_m$$

$$= 2(a-y)\cos\theta_m$$

$$\rightarrow \Delta\psi_{m,OPD} = \frac{2\pi n_{core}}{\lambda_0}(AC - BC) = \frac{4\pi n_{core}}{\lambda_0}(a-y)\cos\theta_m$$

The phase difference due to the single reflection is simply $\Delta\psi_{m,TIR} = -\phi_m$. The total phase difference is then:

$$\Delta\psi_m = \Delta\psi_{m,OPD} + \Delta\psi_{m,TIR}$$

$$= \frac{4\pi n_{core}}{\lambda_0}(a-y)\cos\theta_m - \phi_m$$

To get rid of the θ_m dependence, let's use the waveguide condition:

$$\frac{2\pi n_{core} d}{\lambda_0} \cos\theta_m - \phi_m = m\pi \quad \| \quad d = 2a$$

$$\cos\theta_m = \frac{\lambda_0 (m\pi + \phi_m)}{4\pi n_{core} a}$$

Plugging this into Eq. 1:

$$\Delta\psi_m = \frac{4\pi n_{core}}{\lambda_0}(a - y)\frac{\lambda_0(m\pi + \phi_m)}{4\pi n_{core} a} - \phi_m$$

$$= \frac{(a-y)(m\pi + \phi_m)}{a} - \phi_m$$

$$= m\pi + \phi - \frac{y}{a}(m\pi + \phi_m) - \phi_m$$

$$\Delta\psi_m = m\pi - \frac{y}{a}(m\pi + \phi_m)$$

b) Derive the field variation at C as a function of y (and m, a, ϕ_m) i.e., the wavefunction of the interfering wave in the form $E(y) = E_0 \cos(\omega t + \alpha)$. Note that the solved amplitude E_0 will be a cosine function arising from the geometry, however it has no time-dependence: $E_0 = E_0\big(\psi(y)\big)$. Evidently also the phase shift $\alpha = \alpha\big(\psi(y)\big)$. Note also there will be an arbitrary amplitude scaling coefficient.

Hint: Express the interfering waves as: $E(y) = A\cos(\omega t) + A\cos\big(\omega t + \Delta\psi_m(y)\big)$. You'll probably need the following well-known trigonometric identity: $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$.

First let's write an expression for two interfering waves, using then the trig identity provided:

$$E(y) = A\cos(\omega t) + A\cos\big(\omega t + \Delta\psi_m(y)\big)$$

$$= A\left(2\cos\left(\frac{\omega t + \omega t + \Delta\psi_m}{2}\right)\cos\left(\frac{\omega t - \omega t - \Delta\psi_m}{2}\right)\right)$$

$$= 2A\cos\left(\omega t + \frac{\Delta\psi_m}{2}\right)\cos\left(-\frac{\Delta\psi_m}{2}\right) \quad \| \quad \cos(-A) = \cos A$$

$$= 2A\cos\left(\omega t + \frac{\Delta\psi_m}{2}\right)\cos\left(\frac{\Delta\psi_m}{2}\right)$$

We now have one term that is time-dependent, and one that is not:

$$E(y) = 2A\cos\left(\frac{\Delta\psi_m}{2}\right)\cos\left(\omega t + \frac{\Delta\psi_m}{2}\right)$$

This is already in the form of the wavefunction $E = E_0\cos(\omega t + \alpha)$:

$$E_0 = 2A\cos\left(\frac{\Delta\psi_m}{2}\right) \quad , \quad \alpha = \frac{\Delta\psi_m}{2}$$

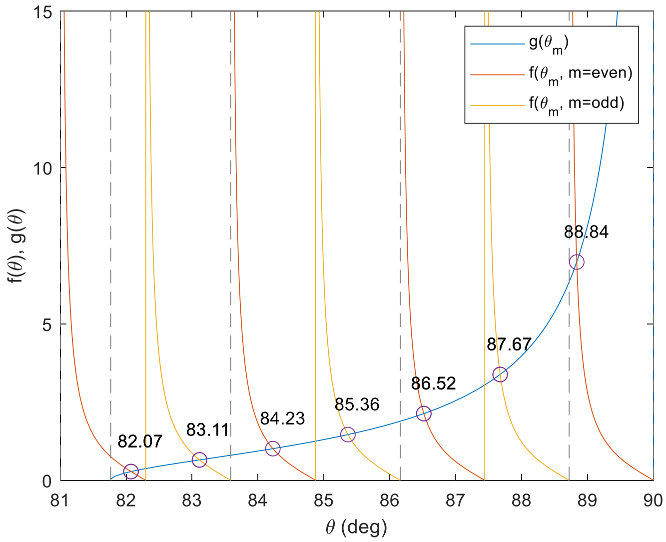
Plugging in $\Delta\psi_m$ from part a):

$$E(y) = 2A\cos\left(\frac{1}{2}\left[m\pi - \frac{y}{a}(m\pi + \phi_m)\right]\right)\cos\left(\omega t + \frac{1}{2}\left[m\pi - \frac{y}{a}(m\pi + \phi_m)\right]\right)$$

c) Plot the wave amplitude $E_0(y)$ for the first three TE modes, given that $a = 10$ µm, $\lambda = 1.3$ µm, $n_{core} = 1.455$ and $n_{cladding} = 1.440$.
Hint: You'll first have to express the phase difference $\Delta\psi$ as in part a). Then find θ_m using the phase difference formula and the waveguide condition, as in problem 2. Solve the phase differences ϕ_m by substituting the solved θ_m values into either equation. The solutions for θ_m will be between 80° and 90°.

As in problem 2, solving graphically for the intercepts of the two functions:

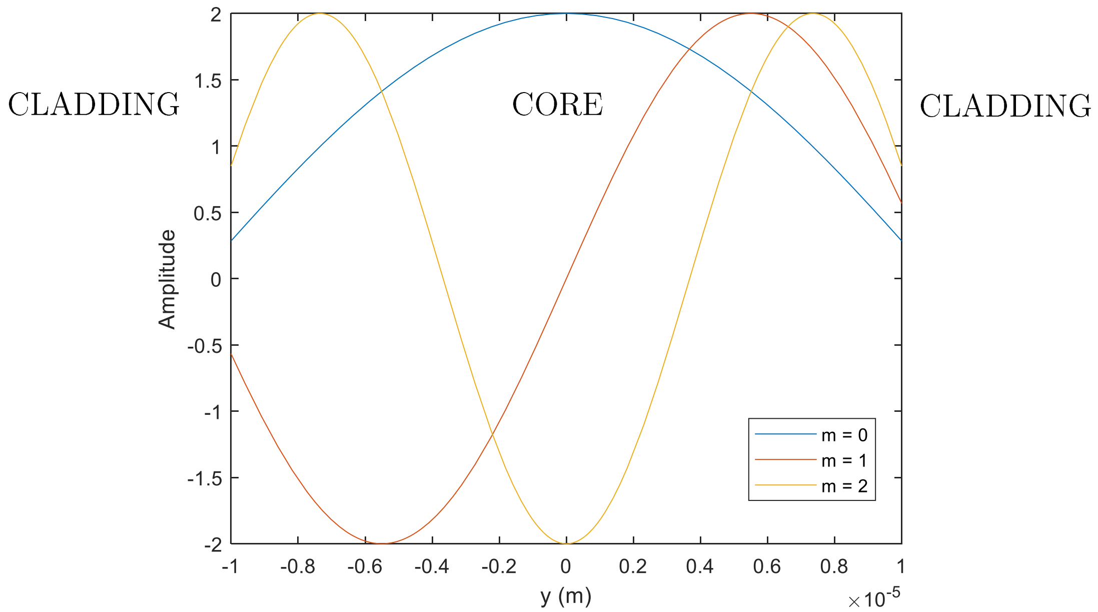
$$f(\theta_m, m) = \tan\left(\frac{\pi n_{core} d}{\lambda_0}\cos\theta_m - \frac{m\pi}{2}\right) \quad , \quad g(\theta_m) = \frac{\sqrt{\sin^2\theta_m - \left(\frac{n_{cladding}}{n_{core}}\right)^2}}{\cos\theta_m}$$



The first three order angles are read from the right: $\theta_0 = 88.84°$, $\theta_1 = 87.67°$, $\theta_2 = 86.52°$
We will solve ϕ_m from the waveguide condition:

$$\frac{2\pi n_{core} d}{\lambda_0}\cos\theta_m - \phi_m = m\pi \rightarrow \phi_m = \frac{2\pi n_{core} d}{\lambda_0}\cos\theta_m - m\pi$$

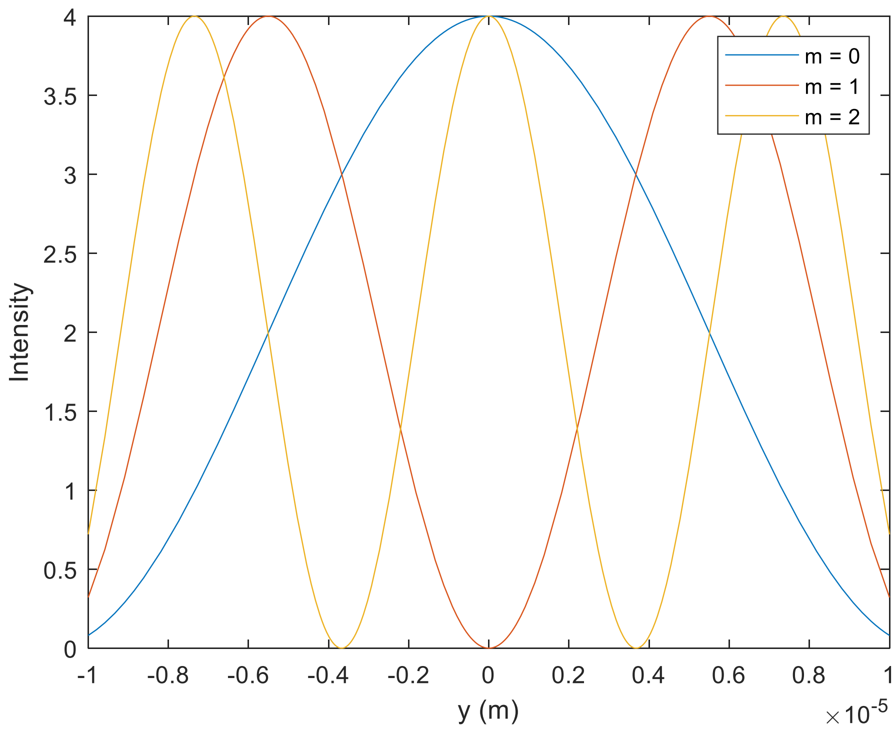
$$\phi_0 = 163.7° \quad , \quad \phi_1 = 147.1° \quad , \quad \phi_2 = 129.8°$$



The plots for E_0 are then:

d) Optical detectors measure intensity profiles. As we saw in Exercise 1, intensity is proportional to the electric field amplitude $I \propto E_0(y)^2$. Plot the intensity profile of the solved electric field.

Taking the square of part c)



https://www.rp-photonics.com/modes.html