Photonics, Spring 2023

Exercises to be returned via Google Classroom by **14:00 next Tuesday 21.2.** For each problem, ½ a point will be awarded for an honest effort and 1 point for a well worked solution. Exercise sessions are held on Tuesdays from 16:00 - 18:00 at Chemicum A121.

Exercise 5, 21.2.2023

1. Bragg diffraction (1 point)

Diffraction gratings can easily be understood by considering a periodic series of slits in an opaque screen (Fig. 1, left). When a plane wave is at normal incidence on the screen, the slits will result in diffracted beams at various angles. Only certain angles will produce constructive interference between beams from neighboring slits, corresponding to a path length difference of integer multiples of the wavelength. This relation is known as the grating equation, or the Bragg diffraction condition: $d \sin\theta = m\lambda;\ m \in \mathbb{Z}$. The value of m dictates the diffraction order.

Consider then a transmission grating where the angle of incidence is θ_i (Fig. 1, middle). Instead of slits, there are periodic scratches on a glass surface. From a straightforward geometric analysis the Bragg diffraction condition must then include the incidence angle:

$$d(\sin\theta_m - \sin\theta_i) = m\lambda$$

Note that although parallel beams are not drawn in the figure, it is from their interference that this condition arises. The treatment is the same for a reflection grating (Fig. 1, right), where the incident and reflected beams are on the same side of the grating. For both cases, the angles of incidence and reflection/refraction are defined to be positive. Note however, that diffractions can produce negative reflection/refraction angles, where the reflected/refracted beam is on the same side as the incident beam.

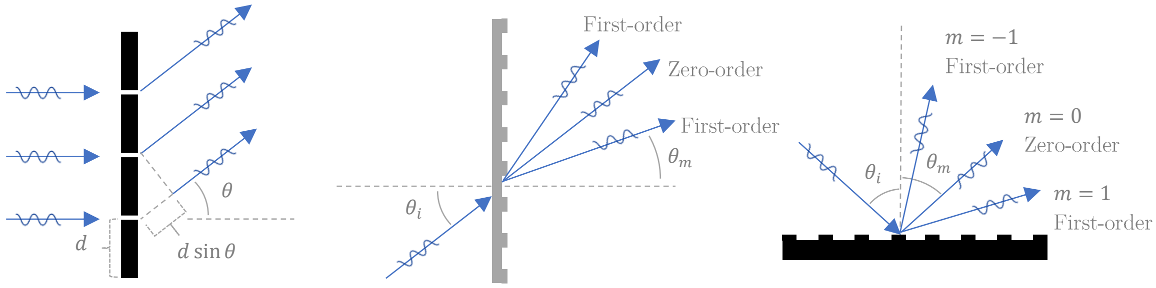


Figure 1: Working principle of a diffraction grating. A plane wave is at normal incidence on periodic slits (left). An incident ray is diffracted at various angles through a transmission grating (middle), and from a reflection grating (right).

Suppose that parallel grooves are etched on the surface of a semiconductor to act as a reflection grating and that the periodicity (separation) of the grooves is 1 µm. If light of wavelength 1.3 µm is incident at an angle 89° to the normal, find the diffracted beams (angles and modes).

Hint: A complex reflection angle implies that an evanescent wave is present. Based on previous exercises, when do these appear? Does a complex angle produce a reflected beam?

Using the grating equation:

$$d(\sin\theta_m - \sin\theta_i) = m\lambda$$

$$\theta_m = \sin^{-1}\left(\frac{m\lambda}{d} + \sin\theta_i\right)$$

A complex angle of reflection would imply an evanescent wave, which is characteristic of TIR i.e., complex angles do not produce a reflected beam.

With $\lambda = 1.3$ µm, $d = 1$ µm, $\theta_i = 89°$:

m	θ_m (°)
-3	Complex
-2	Complex
-1	-17.5
0	89
1	Complex
2	Complex
3	Complex

2. Diffraction grating for WDM (1 point)

Let us now consider an application for a transmission diffraction grating. It would be beneficial to simultaneously send overlayed signals in an optical fiber, that could then be separated at the receiving end. The overlaying of signals could be done based on wavelength; each wavelength component carries its own unique information. By placing a suitable transmission diffraction grating at the receiving end of such a fiber, the wavelength components would separate spatially according to the Bragg diffraction condition (see Problem 1). These spatially separated beams could then be directed into separate detectors. Such a process is called wavelength division (de-)multiplexing (WDM).

Suppose a WDM is used to separate superposed wavelength components. The diffraction grating has a periodicity of 2 µm, and the angle of incidence with respect to the normal of the grating is 0°. What is the angular separation of the two wavelength components at 1550 nm and 1540 nm? How could you increase this separation?

Again using the grating equation, but now $\theta_i = 0 \rightarrow \sin\theta_i = 0$:

$$d \sin\theta_m = m\lambda$$

$$\theta_m = \sin^{-1}\left(\frac{m\lambda}{d}\right)$$

Angular separation:

$$\Delta\theta_m = \theta_m(\lambda_1) - \theta_m(\lambda_2)$$

Calculating each angle:

m	θ_1	θ_2
-3	Complex	Complex
-2	Complex	Complex
-1	-50.805	-50.354
0	0	0
1	50.805	50.354
2	Complex	Complex
3	Complex	Complex

Real values are obtained at $m = 0, \pm 1$. Evidently at $m = 0$ there is no angular separation since the components are superposed. So the angular separations are:

$$\Delta\theta_m = \theta_{m=\pm 1}(\lambda_1) - \theta_{m=\pm 1}(\lambda_2) \approx \pm 0.45°$$

Looking at the argument $m\lambda/d$, better angular separation is achieved by decreasing d. However, $m\lambda$ limits the values of d for real valued angles; $\left|\frac{m\lambda}{d}\right| \leq 1$.

3. Diffraction limit (2 points)

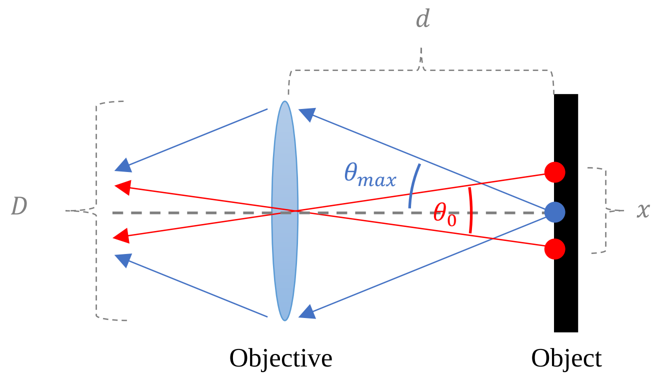
Hint: A great source, with good visual aids:
https://iu.pressbooks.pub/openstaxcollegephysics/chapter/limits-of-resolution-the-rayleigh-criterion/

a) Describe the Rayleigh criterion for the lateral resolution of a microscope.

Rayleigh criterion is defined by the interference of two Airy disk patterns, such that two spots are resolvable when the principal maximum of one diffraction pattern coincides with the first dark ring of the other diffraction pattern. The Airy disk diameter is defined as the diameter of the first dark ring. The Rayleigh criterion is:

$$\sin\theta_0 = 1.22\frac{\lambda}{D}$$

Here D is the diameter of the aperture and θ_0 is the angle between the two rays. An important application of this condition is the minimum resolvable distance for an objective. The numerical aperture NA is a dimensionless number characterizing the range of acceptance angles α of an optical system: $NA = n\sin\theta_{max}$. Considering the geometry of an imaging setup, with the objective parameters in blue and those relating to the Rayleigh criterion in red.



Assuming operation in air ($n \approx 1$) we can write:

$$\sin\theta_{max} = \frac{D}{2d} \quad\rightarrow\quad NA = \frac{nD}{2d} \approx \frac{D}{2d} \quad\rightarrow\quad D = 2dNA$$

Combining this with the Rayleigh criterion, we can write the **smallest resolvable distance**:

$$\sin\theta_0 = \frac{x}{d} = 1.22\frac{\lambda}{D} \quad\rightarrow\quad x = 1.22\frac{\lambda d}{2dNA}$$

$$\boldsymbol{x = 0.61\frac{\lambda}{NA}}$$

b) Calculate the size of the smallest resolvable feature for a 50x objective at wavelength 250 nm, 350 nm, and 750 nm. Assume operation in air ($n \approx 1$), and a maximum acceptance angle of 30°. What would you expect to see from a Blue-ray disc?

Let's first calculate the numerical aperture:

$$NA = n\sin\theta_{max} = \sin 30° = 0.5$$

The smallest resolvable distance is given by:

$$x = 0.61\frac{\lambda}{NA}$$

For the various wavelengths:

$$x(\lambda = 250\text{ nm}) = 305\text{ nm} \quad , \quad x(\lambda = 350\text{ nm}) = 427\text{ nm} \quad , \quad x(\lambda = 750\text{ nm}) \approx 915\text{ nm}$$

The separation of grooves in a Blu-ray disc is ~ 320 nm, so only the 250 nm source would be able to resolve the grooves.

c) “Super-resolution” imaging techniques break the diffraction limit (Rayleigh criterion / Abbe limit). What are two assumptions of the Rayleigh criterion i.e., when is it valid?

   Hint: One has to do with the geometry of the optical setup, the other with the constituent material properties.

One assumption is that the imaging detector is in the far-field, which inhibits measuring evanescent waves. One super-resolution approach is to bring the detector close enough to the subject to measure evanescent waves.

Another assumption of the Rayleigh limit is that the optical system comprises linear optical materials i.e., materials in which radiation-induced polarization is directly proportional to the amplitude of the radiation. Far-field super-resolution techniques utilize materials that display optical nonlinearity in the reflected light.

4. **Fabry-Perot optical cavity** (4 points)

A Fabry-Perot optical cavity or resonator is an optical device that can store radiation energy at certain frequencies. The cavity is formed between two perfectly aligned identical mirrors separated by a distance. When light is introduced into the cavity, the light undergoes a series of subsequent reflections at each mirror. Evidently for the waves to remain within the cavity, they must interfere constructively to form a standing wave. Considering that the mirror surface is metallic, the electric field at the surface must be zero, which implies that only integer multiples of half-wavelength can fit into the cavity.

There are two important quantities that describe the performance of the cavity. Finesse is a unitless quantity that measures how narrow (in frequency) the resonant modes are with respect to their separation in frequency. A higher finesse implies that the modes are well separated, whereas a low finesse implies that the modes begin to overlap. This quantity is intrinsically related to the reflectance of the constituent mirrors; mirrors with $R < 1$ introduce some phase distortions, resulting in a broadened spectra of permissible frequencies about the central modes. In general Finesse is defined for interfering waves:

$$F = \frac{\delta f}{f}$$

δf is the spacing between modes, and f is the mode spectral width. For optical cavities, Finesse can be written as a function of reflectance R:

$$F = \frac{\pi\sqrt{R}}{1 - R}$$

The other relevant quantity is quality factor Q, which is a unitless quantity describing how fast the resonant modes diminish inside the cavity. The Q-factor also indicates how selective the resonator is in frequency; the higher the Q-factor, the narrower the spectral width of the resonant modes are. Naturally the Q-factor is closely related to the Finesse, and is formally defined as:

$$Q = \frac{f_m}{\delta f} = mF$$

f_m is the particular mode frequency and m is the mode number.

Consider a Fabry-Perot cavity designed to operate at $\lambda = 632.8$ nm, with a mirror separation of 50 cm, and a medium refractive index of 1. The mirror reflectances are 0.97 each.

a) What is the nearest mode number that corresponds to a radiation of wavelength 632.8 nm?

Only integer multiples of half-wavelength can exist in the cavity:

$$L = m\frac{\lambda}{2} \to m = \frac{2L}{\lambda} = \frac{2 * 50\ \text{cm}}{632.8\ \text{nm}} \approx 1\,580\,278$$

b) What is the actual wavelength of the mode closest to 632.8 nm?

$$\lambda = \frac{2L}{m} = \frac{2 * 50\ \text{cm}}{1\,580\,278} \approx 632.800052\ \text{nm}$$

c) What is the mode separation in frequency?

Hint: $f = c/\lambda$)

The frequency separation is simply the separation between modes m and $m + 1$:

$$\Delta f_m = f_{m+1} - f_m = \frac{c}{\lambda_{m+1}} - \frac{c}{\lambda_m} = \frac{(m+1)c}{2L} - \frac{mc}{2L} = \frac{c}{2L}(m + 1 - m) = \frac{c}{2L} \approx 299.8\ \text{MHz}$$

d) What are the Finesse F and Q-factors for the cavity?

Hint: The Finesse is independent of frequency, while the Q-factor is not. For calculating the Q-factor, use the results from part a.

Finesse:

$$F = \frac{\pi\sqrt{R}}{1-R} = \frac{\pi\sqrt{0.97}}{1-0.97} \approx 103$$

Q-factor:

$$Q = \frac{f_m}{\delta f_m} = mF \approx 1.58 * 10^6 * 103 \approx 1.63 * 10^8$$