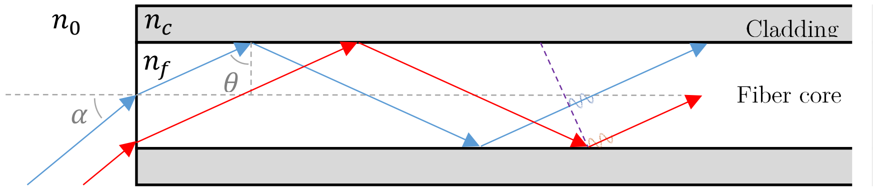
Photonics, Spring 2023

Exercises to be returned via Google Classroom by **14:00 next Tuesday 14.2.** For each problem, ½ a point will be awarded for an honest effort and 1 point for a well worked solution. Exercise sessions are held on Tuesdays from 16:00 - 18:00 at Chemicum A121.

Exercise 4, 14.2.2023

1. Optical fiber dimensions (3 points)

Optical fibers act as waveguides, promoting the propagation of certain modes of EM radiation.



Consider the simple case of two parallel waves incident on an optical fiber. For the propagating waves to not cancel out, constructive interference must occur; the two rays must remain in phase throughout reflections. The mathematical analysis of this condition results in the waveguide condition:

$$\left[\frac{2\pi}{\lambda}\right] d\cos\theta - \phi(\theta) = m\pi \quad m \in \mathbb{Z}$$

This condition tells us that only certain reflection angles are permitted. For a certain wavelength, the waves corresponding to the permitted angles are referred to as wave modes. Evidently the upper limit of possible angles is imposed by the condition of TIR. With a sufficiently small fiber core diameter only the lowest mode ($m = 0$, $\theta \approx 90°$) will propagate. Such fibers are appropriately called single-mode fibers.

There are a few important characteristic quantities describing an optical fiber. The numerical aperture NA is a dimensionless number characterizing the range of acceptance angles α of an optical system:

$$NA = n_0 \sin\alpha_{max}$$

The refractive indices of the cladding and core must evidently differ for reflections to occur. The normalized index difference is then defined in terms of the cladding and fiber core:

$$\Delta = \frac{n_f - n_c}{n_f}$$

Finally, an important characteristic parameter is the V-number, which is related to the number of propagating modes. The parameter arises from the Maxwellian analysis of EM intensity fields in cylindrical waveguides, namely Bessel functions. It depicts a fundamental relationship between numerical aperture, operating vacuum wavelength, and the core diameter:

$$V_{number} = \frac{\pi D}{\lambda_0} NA, \text{ For single-mode fibers: } V_{number} \leq 2.405$$

A typical single mode fiber has a core diameter of 8 µm and a refractive index of $n_f = 1.46$. The normalized index difference is 0.3%, and the cladding diameter 125 µm. Calculate the following:

a) Numerical aperture.
Hint: For rays of a certain angle α to propagate in the fiber TIR must occur. What is the condition for TIR in terms of θ? Using Snell's law express NA as a function of n_c and n_f. Note that:

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x, \quad \cos\left[\operatorname{asin}\left(\frac{a}{b}\right)\right] = \sqrt{\frac{b^2 - a^2}{b^2}}$$

$$n_0 \sin \alpha_{max} = n_f \sin\left(\frac{\pi}{2} - \theta_c\right) = n_f \cos \theta_c \quad 1$$

$$n_f \sin \theta_c = n_c \sin \frac{\pi}{2} = n_c \rightarrow \theta_c = \sin^{-1} \frac{n_c}{n_f} \quad 2$$

Substituting Eq. 2 to Eq. 1:

$$NA = n_0 \sin \alpha_{max} = n_f \cos\left(\left(\sin^{-1} \frac{n_c}{n_f}\right)\right) = n_f \sqrt{\frac{n_f^2 - n_c^2}{n_f^2}} = \sqrt{n_f^2 - n_c^2} \quad 3$$

Now solving for n_c:

$$\Delta = \frac{n_f - n_c}{n_f} \rightarrow n_c = n_f(1 - \Delta)$$

$$n_c = 1.46(1 - 0.003) = 1.45562$$

Finally solving Eq. 3:

$$NA = \sqrt{n_f^2 - n_c^2} = \sqrt{1.46^2 - 1.45562^2} \approx 0.113$$

b) Acceptance angle α_{max} (assume $n_0 = n_{air} \approx 1$).

From Eq. 3:

$$NA = n_0 \sin \alpha_{max}$$

$$\alpha_{max} = \sin^{-1}\left(\frac{NA}{n_0}\right) = \sin^{-1}(NA) \approx 6.5°$$

c) Cut-off wavelength.

For a single-mode fiber $V_{number} \leq 2.405$, which will give us the minimum wavelength for producing single-mode propagation.

$$V_{number} = \frac{\pi D}{\lambda_0} NA$$

$$V_{number} \leq 2.405$$

$$2.405 \geq \frac{\pi D}{\lambda_0} NA$$

$$\lambda \geq \frac{\pi D}{2.405} NA = \frac{\pi 8 \text{ µm}}{2.405} \sqrt{1.46^2 - 1.45562^2} \approx 1.18 \text{ µm}$$

It is worth noting that while smaller wavelengths can propagate in the fiber, they will manifest multiple modes.

2. **Group index** (2 points)

The refractive index is defined as the ratio of vacuum speed of light and the phase velocity in the material. While in vacuum or air the group and phase velocities are equivalent, this is not necessarily the case in other media. Group velocity, which determines the speed of energy propagation, is defined as:

$$v_g = \frac{d\omega}{dk}$$

For a media where $n = n(\lambda)$, we can define an analogous group refractive index as the ratio of vacuum speed and group velocity:

$$N_g = \frac{c}{v_g}$$

The wavelength-dependent refractive index can be modeled using the Sellmeier equation:

$$n^2 = 1 + \frac{A_1\lambda^2}{\lambda^2 - \lambda_1^2} + \frac{A_2\lambda^2}{\lambda^2 - \lambda_2^2} + \frac{A_3\lambda^2}{\lambda^2 - \lambda_3^2} + \cdots$$

For pure silica, the first three coefficients are:

$$A_1 = 0.696749 \quad A_2 = 0.408218 \quad A_3 = 0.890815$$

$$\lambda_1 = 0.0690660\ \mu\text{m} \quad \lambda_1 = 0.115662\ \mu\text{m} \quad \lambda_1 = 9.900559\ \mu\text{m}$$

a) Find an expression for the group index N_g in terms of $n, \lambda, dn/d\lambda$.

Hint: Express k, ω as functions of vacuum wavelength λ_0, then use the chain rule and the inverse function rule. Notice that since $n(\lambda_0)$, you will also need to apply the product rule.

First, we define useful equations:

$$v_g = \frac{d\omega}{dk} \qquad k = \frac{2\pi n}{\lambda_0} \qquad \omega = 2\pi f = \frac{2\pi c}{\lambda_0}$$

Using the chain rule:

$$v_g = \frac{d\omega}{dk} = \frac{d\omega}{d\lambda_0}\frac{d\lambda_0}{dk} = \frac{d}{d\lambda_0}\left(\frac{2\pi c}{\lambda_0}\right)\frac{d}{dk}\left(\frac{2\pi}{\lambda_0}\right)$$

Applying the inverse function rule:

$$v_g = \frac{d\omega}{dk} = \frac{d\omega}{d\lambda_0}\left(\frac{dk}{d\lambda_0}\right)^{-1} = \frac{d}{d\lambda_0}\left(\frac{2\pi c}{\lambda_0}\right)\left(\frac{d}{d\lambda_0}\left[\frac{2\pi n}{\lambda_0}\right]\right)^{-1}$$

Applying the product rule to the second term:

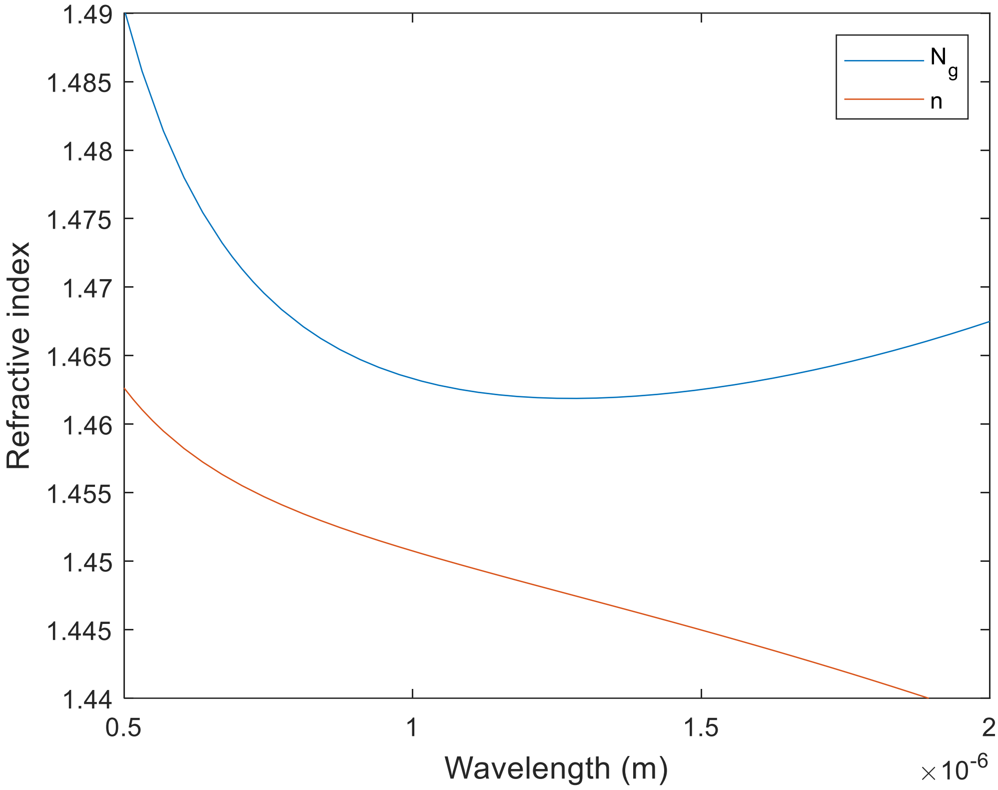
$$v_g = -\frac{2\pi c}{{\lambda_0}^2}\left(\frac{d}{d\lambda_0}\left[2\pi n * \frac{1}{\lambda_0}\right]\right)^{-1} = -\frac{2\pi c}{{\lambda_0}^2}\left(\frac{1}{\lambda_0} * \frac{d(2\pi n)}{d\lambda_0} + 2\pi n\frac{d}{d\lambda_0}\left(\frac{1}{\lambda_0}\right)\right)^{-1}$$

$$v_g = -\frac{2\pi c}{{\lambda_0}^2}\left(\frac{2\pi}{\lambda_0}\frac{dn}{d\lambda_0} - \frac{2\pi n}{\lambda_0^2}\right)^{-1} = -\frac{2\pi c}{{\lambda_0}^2}\frac{1}{\frac{2\pi}{\lambda_0}\frac{dn}{d\lambda_0} - \frac{2\pi n}{\lambda_0^2}}$$

$$v_g = \frac{c}{n - \lambda\frac{dn}{d\lambda_0}} = \frac{c}{N_g} \rightarrow N_g = n - \lambda\frac{dn}{d\lambda}$$

b) Calculate the group index N_g and refractive index n numerically from 500 nm to 1.8 µm.
Hint: Use e.g. Python or MATLAB to solve the derivative in N_g, then plot n and N_g as functions of wavelength.

Plotted with MATLAB:



3. **Material dispersion** (1 point)

Calculate the temporal broadening of an optical pulse in a 1 km long pure silica fiber due to material dispersion for an LED operating at 850 nm (linewidth 20 nm).

Hint: Based on the pervious exercise, what is the time of propagation of energy for the pulse? Can you express this as a function of N_g? How much does the time of propagation then change as a function of wavelength? You should get an expression with a second order derivative, which you can solve numerically with parameters from Exercise 2. Note that since the time spread and linewidth are much smaller than the propagation time and wavelength, we can approximate $\frac{dt}{d\lambda} \approx \frac{\Delta t}{\Delta\lambda}$.

The time of propagation (of energy) is simply the ratio of length and group velocity:

$$t = \frac{L}{v_g} = \frac{LN_g}{c} = \frac{L}{c}\left(n - \lambda\frac{dn}{d\lambda}\right)$$

$$\frac{dt}{d\lambda} = \frac{L}{c}\left(\frac{dn}{d\lambda} - \frac{d}{d\lambda}\lambda\frac{dn}{d\lambda}\right)$$

Using the product rule on the third term:

$$\frac{dt}{d\lambda} = \frac{L}{c}\left(\frac{dn}{d\lambda} - \lambda\frac{d^2n}{d\lambda^2} + \frac{dn}{d\lambda}\frac{d\lambda}{d\lambda}\right) = \frac{L}{c}\left(\frac{dn}{d\lambda} - \lambda\frac{d^2n}{d\lambda^2} + \frac{dn}{d\lambda}\right)$$

$$\frac{dt}{d\lambda} = -\frac{L}{c}\lambda\frac{d^2n}{d\lambda^2}$$

Since the time spread and linewidth are much smaller than the propagation time and wavelength, we can approximate $\frac{dt}{d\lambda} \approx \frac{\Delta t}{\Delta\lambda}$:

$$\Delta t = \left|\frac{dt}{d\lambda}\right|\Delta\lambda = \left|-\frac{L}{c}\lambda\frac{d^2n}{d\lambda^2}\right|\Delta\lambda$$

From code written for previous exercise:

$$\frac{d^2n}{d\lambda^2}(\lambda = 850\text{ nm}) \approx 29727145756$$

$$\Delta t = \left|-\frac{1000\text{m}}{3 * 10^8\text{ms}^{-1}} 850 * 10^{-9}\text{m} * 29727145756\text{m}^{-2}\right| 20 * 10^{-9}\text{m} \approx 1.68\text{ ns}$$

4. **Attenuation & scattering** (3 point)

a) 1 mW of optical power enters a single mode fiber from one end. A photodetector at the opposite end of the fiber has a detection limit of 10 nW. Assuming that the signal is barely detected after 130 km of fiber, compute the attenuation coefficient (decibels / length).

Hint: Power (and intensity) decays exponentially as a function of distance: $P_{out} = P_{in}\exp(-\alpha L)$.

To convert attenuation to decibels per kilometer: $\alpha_{dB} = \frac{10}{\ln 10}\alpha$

$$P_{out} = P_{in}e^{-\alpha L}$$

$$\alpha = -\frac{1}{L}\ln\frac{P_{out}}{P_{in}}$$

$$\alpha = \frac{1}{L}\ln\frac{P_{in}}{P_{out}} \quad \| \quad -\ln\frac{a}{b} = \ln\frac{b}{a}$$

$$\alpha = \frac{1}{130\,000\text{ m}}\ln\frac{1*10^{-3}\text{W}}{10*10^{-9}\text{W}} \approx 8.856*10^{-5}\text{m}^{-1}$$

$$\alpha_{dB} = \frac{10}{\ln 10}\alpha = \frac{10}{\ln 10}8.856*10^{-5}\text{m}^{-1} \approx 0.385\frac{\text{dB}}{\text{km}}$$

b) The irradiance of oscillating dipoles (as is the case in Rayleigh scattering) is proportional to $1/\lambda^4$.

Attenuation in glass fiber due to Rayleigh scattering is approximately given by:

$$\alpha_R \approx \frac{8\pi^3}{3\lambda^4}(n^2-1)^2\beta_T k_B T_f$$

λ_0 is the vacuum wavelength, $n(\lambda)$ is the refractive index, β_T is the isothermal compressibility (at T_f) and k_B is the Boltzmann coefficient. T_f is the fictive, or glass transition temperature, at which a phase transition occurs during cooling from liquid to glass. As such, T_f indicates the temperature below which density is approximately constant.

i) Explain briefly how Rayleigh scattering attenuates the signal.

Rayleigh scattering is generally the scattering from particles with sizes far below the wavelength of radiation. In a material such scatterers correspond to inhomogeneities (e.g. defects) with a refractive index differing from the bulk. These inhomogeneous regions act as dielectric particles, that scatter propagating light into various directions, causing the apparent attenuation of the wave.

ii) Calculate α_R for silica at $\lambda = 1.55\ \mu\text{m}$ (Check literature for coefficient values). The experimental values of attenuation around this wavelength is about 0.2 dB km^{-1}. What can you conclude?

Values from google:

Wavelength $\lambda = 1.55\ \mu\text{m}$

Boltzmann constant $k_B = 1.38*\frac{10^{-23}\text{J}}{\text{K}}$

Refractive index $n(\lambda = 1.55\ \mu\text{m}) = 1.444$

Fictive temperature $T_f = 1480\text{ K}$

Isothermal compressibility $\beta_T(T_f = 1480\text{ K}) = 7*10^{-11}\frac{\text{m}^2}{\text{N}}$

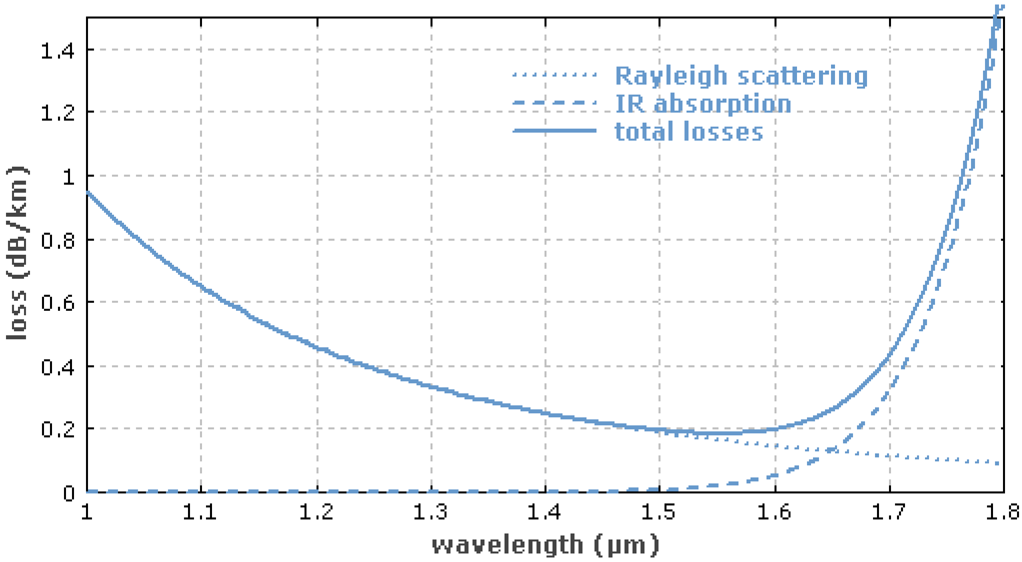
$$\alpha_R \approx \frac{8\pi^3}{3\lambda^4}(n^2-1)^2\beta_T k_B T_f$$

$$\alpha_R \approx \frac{8\pi^3}{3(1.55\mu\text{m})^4}(1.444^2 - 1)^2 7 * 10^{-11}\frac{\text{m}^2}{\text{N}} * 1.38 * \frac{10^{-23}\text{J}}{\text{K}} * 1480\text{ K}$$

$$\alpha_R \approx 2.412 * 10^{-5}\text{m}^{-1} = 2.412 * 10^{-2}\text{km}^{-1}$$

$$\alpha_{R,dB} = \frac{10}{\ln 10}\alpha_R \approx 1.05 * 10^{-4}\frac{\text{dB}}{\text{m}} \approx 0.105\frac{\text{dB}}{\text{km}}$$

Comparing to experimental values, there must be an additional contribution to attenuation. This additional attenuation arises from the molecular absorption of IR radiation.



Intrinsic Losses of Silica Fiber