Photonics, Spring 2023

Exercises to be returned via Google Classroom by **14:00 next Tuesday 7.2.** For each problem, ½ a point will be awarded for an honest effort and 1 point for a well worked solution. This week the maximum points are 8, with the possibility for 2 extra points from the bonus exercise.

Exercise sessions are held on Tuesdays from 16:00 - 18:00 at Chemicum A121.

Exercise 2, 7.2.2023

1. Coherence lengths in terms of linewidth (X points)

All real light sources (including monochromatic) display a range of frequencies around the central frequency. The phenomenon arises from electron transitions that produce EM waves. Electron transmissions have a duration of the order of 10^{-8}s to 10^{-9}s, which manifests a spread in the frequencies produced. Also, due to the thermal motion of atoms, the frequency spread is further altered by the Doppler effect. Finally, atoms undergo collisions that inhibit wavetrains, further broadening the frequency distribution. Due to the spread of frequencies, propagating EM waves (from real light sources) have a finite distance (coherence length) after which the wave no longer behaves as a simple sinusoidal wave, and phase can no longer be predicted reliably. The frequency (wavelength) spread is known as the bandwidth $\Delta\nu$ (linewidth $\Delta\lambda_0$), defined as the full-width half-maximum (FWHM) of the intensity spectrum. Just as waves can be described temporally (frequency) or spatially (wavelength), coherence can be described through analogous terms; temporal and spatial coherence.

a) Consider a light source with a frequency bandwidth $\Delta\nu$ and center frequency ν_0. Derive expressions for the coherence length and the corresponding coherence time of a wave in terms of the linewidth $\Delta\lambda_0$.

Hint: Since the linewidth and bandwidth are much smaller than the wavelength and frequency, the approximation can be made: $\frac{\Delta\nu}{\Delta\lambda} \approx \left|\frac{\partial\nu}{\partial\lambda_0}\right|$.

Starting from the defining frequency as a function of speed and wavelength:

$$c = \nu\lambda_0 \rightarrow \nu = \frac{c}{\lambda_0} \qquad 1$$

Differentiating Eq. 1 with respect to wavelength:

$$\frac{\partial\nu}{\partial\lambda_0} = -\frac{c}{\lambda_0^2}$$

Since the linewidth and bandwidth are much smaller than the wavelength and frequency, we can approximate:

$$\frac{\Delta\nu}{\Delta\lambda} \approx \left|\frac{\partial\nu}{\partial\lambda_0}\right| = \frac{c}{\lambda_0^2}$$

$$\Delta\nu = \frac{c}{\lambda_0^2}\Delta\lambda_0 \quad , \quad \Delta\lambda_0 = \frac{\lambda_0^2}{c}\Delta\nu \qquad 2,3$$

The temporal coherence Δt_c can be expressed as a function of $\Delta\nu$:

$$\Delta t_c = \frac{1}{\Delta\nu} \quad \| \quad Subst.Eq.2$$

$$\Delta t_c = \frac{\lambda_0^2}{c\Delta\lambda_0} \quad , \quad \Delta\lambda_0 = \frac{\lambda_0^2}{c\Delta t_c} \qquad 4,5$$

The temporal coherence can be converted to coherence length:

$$c = \frac{l_c}{\Delta t_0} \quad \rightarrow \quad l_c = c\Delta t_c = c\frac{1}{\frac{c}{\lambda_0^2}\Delta\lambda_0} = \frac{\lambda_0^2}{\Delta\lambda_0} \qquad 6$$

b) Estimate the coherence time and length of a white light in vacuum from the spectral width, given the usual limits $\lambda_{UV} \leq 350 \text{ nm}, \lambda_{\text{IR}} \geq 750 \text{ nm}$.

Given the limits, the linewidth for white light is:

$$\Delta\lambda_0 = 350 \text{ nm} - 750 \text{ nm} = 400 \text{ nm}$$

The center frequency is:

$$\lambda_0 = \frac{350 \text{ nm} + 750 \text{ nm}}{2} = 550 \text{ nm}$$

With Eq. 6:

$$l_c = \frac{\lambda_0^2}{\Delta\lambda_0} = \frac{(550 \text{ nm})^2}{400 \text{ nm}} \approx 756 \text{ nm}$$

With Eq. 4:

$$\Delta t_c = \frac{\lambda_0^2}{c\Delta\lambda_0} = \frac{1}{c} l_c = \frac{1}{3 * 10^8 \text{ ms}^{-1}} 756 \text{ nm} \approx 2.52 \text{ fs}$$

2. Coherence length of different light sources (X points)

Find the coherence length of the following light sources in vacuum.

a) A LED emitting at 1550 nm with a spectral width of 150 nm, a semiconductor laser diode emitting at 1550 nm with a spectral width of 3 nm, and a quantum well semiconductor laser diode emitting at 1550 nm with a spectral width of 0.1 nm.

LED:

$$l_c = \frac{\lambda_0^2}{\Delta\lambda_0} = \frac{(1550\ \text{nm})^2}{150\ \text{nm}} \approx 16\ \mu\text{m}$$

Semiconductor laser diode:

$$l_c = \frac{\lambda_0^2}{\Delta\lambda_0} = \frac{(1550\ \text{nm})^2}{3\ \text{nm}} \approx 801\ \mu\text{m}$$

Quantum well semiconductor laser diode:

$$l_c = \frac{\lambda_0^2}{\Delta\lambda_0} = \frac{(1550\ \text{nm})^2}{0.1\ \text{nm}} \approx 24\ \text{mm}$$

b) A multimode HeNe laser with a spectral frequency width of 1.5 GHz, and a specially designed single mode and stabilized HeNe laser with a spectral width of 100 MHz.

Multimode HeNe laser with a spectral frequency width of 1.5 GHz.

$$l_c = \frac{\lambda_0^2}{\Delta\lambda_0} = \frac{\lambda_0^2}{\frac{\lambda_0^2}{c}\Delta\nu} = \frac{c}{\Delta\nu} \quad \| \quad \text{Subst.}\, \Delta\lambda_0 = \frac{\lambda_0^2}{c}\Delta\nu$$

$$l_c = \frac{3 * 10^8 \text{ms}^{-1}}{1.5\ \text{GHz}} \approx 0.2\ \text{m}$$

Specially designed single mode and stabilized HeNe laser:

$$l_c = \frac{3 * 10^8 \text{ms}^{-1}}{100\ \text{MHz}} = 3\ \text{m}$$

3. Interference fringes: Michelson interferometer example (3 points)

The fringes that appear on the detector of a Michelson interferometer arise from the interference of split rays with differing optical path lengths (OPLs). Rays propagating from the source will diverge in space, occupying a larger surface area on the detector. Furthermore, rays will have optical path differences (OPDs) depending on how much they diverge form the optical axis. A lens is placed in front of the detector to converge parallel split rays to a single point on the detector. As a result, moving one of the mirrors will bring about an OPD between these parallel split rays giving rise to interference upon convergence. For the beam-splitter $R = T = 0.5$.

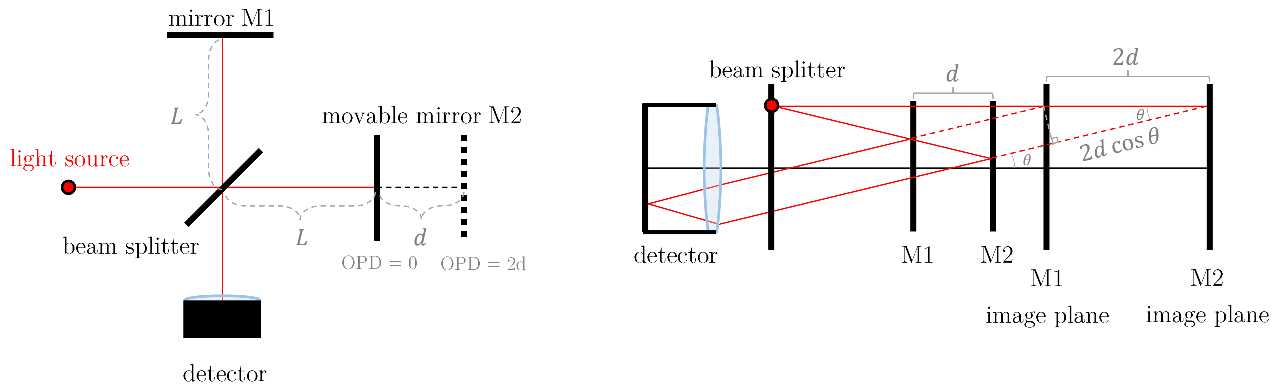

Fig 1: A) Schematic (left) and analogous ray diagram (right) of Michelson interferometer. Note that in the analogous diagram, the source is shifted above the optical axis of the detector to ease visualization. The OPL $\approx 2dcos\,\theta$.

a) Derive an expression for the half-width γ (full angular width at intensity half-maximum) of the fringes.
Hint: For two interfering waves, the total intensity can be expressed as a sum of each component's intensities and the intensity arising from interference. This is known as a coherence function:

$$I_{tot} = \frac{1}{\mu_0 c}\langle \boldsymbol{E_{tot}} \cdot \boldsymbol{E_{tot}} \rangle = \frac{1}{\mu_0 c}\langle (\boldsymbol{E_1} + \boldsymbol{E_2}) \cdot (\boldsymbol{E_1} + \boldsymbol{E_2}) \rangle = \frac{1}{\mu_0 c}(\langle \boldsymbol{E_1} \cdot \boldsymbol{E_1} \rangle + \langle \boldsymbol{E_2} \cdot \boldsymbol{E_2} \rangle + 2\langle \boldsymbol{E_1} \cdot \boldsymbol{E_2} \rangle)$$

$$\rightarrow I_{tot} = I_1 + I_2 + I_{12} = I_1 + I_2 + \frac{2}{\mu_0 c}\langle \boldsymbol{E_1} \cdot \boldsymbol{E_2} \rangle$$

It is worth noting that the first two terms are DC components, and the third the interference component. Each wave is defined as $\boldsymbol{E_n} = \boldsymbol{E_{n0}} \sin(\omega t - \phi_{\mathrm{n}})$, where ω is angular frequency and ϕ is the relative phase-shift. It is enough to evaluate the waves in 1D (parallel planewaves). What should be the integration limit, i.e., what is the period of the function $\sin \omega t * \sin(\omega t + a)$? You can check for instance on an online graphical calculator.

The period for a $\sin a * \sin(a + b)$ is $T = \frac{\pi}{\omega}$, serving as the integration limit:

$$I_{12} = \frac{2}{\mu_0 c}\langle E_1 E_2 \rangle = \frac{2}{\mu_0 c}\frac{\int_0^{\frac{\pi}{\omega}} E_1 E_2 \sin(\omega t) * \sin(\omega t + \phi)\, dt}{T}$$

$$= \frac{2}{\mu_0 c}\frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} E_1 E_2 \sin(\omega t) * \sin(\omega t + \phi)\, dt$$

$$= \frac{2}{\mu_0 c}\frac{\omega}{\pi} E_1 E_2 \int_0^{\frac{2\pi}{\omega}} \sin(\omega t) * \sin(\omega t + \phi)\, dt$$

$$= \frac{1}{\mu_0 c} E_1 E_2 \cos\phi$$

$$I = I_1 + I_2 + I_{12} = I_1 + I_2 + \frac{1}{\mu_0 c} E_1 E_2 \cos\phi = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\phi$$

Now $I_1 = I_2 = I_0$

$$I = 2I_0(1 + \cos\phi)$$

And the maximum intensity $\cos\phi = 1 \rightarrow \phi_{max} = 2\pi n$:

$$I_{max} = 4I_0$$

At half width (full width, half maximum), the irradiance is by definition at half of the maximum irradiance:

$$I_{hm} = \frac{I_{max}}{2} = \frac{4I_0}{2} = 2I_0(1 + \cos\phi_{hm})$$

$$\phi_{hm} = 2\pi n - \frac{\pi}{2} \quad n \in \mathbb{Z}$$

The half width is then:

$$\gamma = 2(\phi_{max} - \phi_{hm}) = \pi$$

b) What is the phase separation between adjacent maxima.
Hint: This should be straightforward from (a).

$$\phi_{sepp} = \phi_{max,n+1} - \phi_{max,n} = 2\pi(n+1) - 2\pi n = 2\pi$$

a) The relationship between mirror displacement d and phase shift in the fringes ϕ allows for precise displacement measurements. As the mirror is displaced, the fringes appear move through space; by observing a stationary point on the interference pattern, the number of fringes passing that point over the duration of the displacement is related to the mirror displacement. How much does the mirror need to be displaced for the fringes to move by one period?
Hint: Look at the ray-diagram and think about how the relative phases of the waves change as the mirror is moved. The answer is simple.

A displacement of λ corresponds to a phase change of 2π.

4. Coherence length: Michelson interferometer example (1 points)

A Michelson interferometer is illuminated by red cadmium light with a mean wavelength of 643.837 nm and a linewidth of 0.0013 nm. The initial setting is for zero optical path difference (O.P.D.) i.e., d = 0. By how much must one of the mirrors be shifted for the fringes to disappear? How many wavelengths does this correspond to?

Hint: An important distinction to understand the Michelson interferometer is related to the coherence length. For a single source, coherence length was defined (Exercise 1) as the distance after which dispersion has deformed the wave far from its original sinusoidal form. Although the wave is deformed, it comprises a well-defined superposition of frequency components. Therefore, the observable interference of waves is not limited to distances below the coherence length. The limit of observable interference of two identical sources is rather limited by the separation of the sources; beyond a separation corresponding to the coherence length l_c, the two waves are no longer coherent i.e., their interference fringes disappear. In other words, at periodic distances (corresponding to a beat frequency) the different frequency components will again be in phase for the duration of the coherence length.

Beyond the coherence length l_c, the two waves are no longer coherent i.e., the fringes disappear:

$$2d = l_c$$

$$d = \frac{l_c}{2} = \frac{\lambda_0^2}{2\Delta\lambda_0} \approx 15.9 \text{ cm}$$

In wavelengths:

$$\frac{d}{\lambda} \approx 247\,000$$

BONUS: Coherence function, continuation of problem 3 (X points)

Assume that the spectrum of the light source in problem 3 has a Gaussian shape. For two interfering waves, the total intensity can be expressed as a sum of each component's intensities and the intensity arising from interference. This is known as a coherence function:

$$I_{tot} = \frac{1}{\mu_0 c}\langle \boldsymbol{E_{tot}} \cdot \boldsymbol{E_{tot}} \rangle = \frac{1}{2\mu_0 c}\langle (\boldsymbol{E_1} + \boldsymbol{E_2}) \cdot (\boldsymbol{E_1} + \boldsymbol{E_2}) \rangle = \frac{1}{\mu_0 c}(\langle \boldsymbol{E_1} \rangle + \langle \boldsymbol{E_1} \rangle + 2\langle \boldsymbol{E_1} \cdot \boldsymbol{E_2} \rangle)$$

$$\rightarrow I_{tot} = I_1 + I_2 + I_{12} = I_1 + I_2 + \frac{2}{\mu_0 c}\langle \boldsymbol{E_1} \cdot \boldsymbol{E_2} \rangle$$

Each wave is defined as $\boldsymbol{E_n} = \boldsymbol{E_{n0}} \sin(\omega t - k_n x_n - \phi_n)$, where ω is angular frequency, k is wavenumber, x is position and ϕ is the relative phase-shift. The relative phase-shift between two waves of equal frequency can also be expressed as a function of their spatial separation:

$$\phi = \frac{2\pi}{\lambda_0}(x_2 - x_1) = k\Delta x$$

Spatial separation in our case refers to the OPL difference of waves arriving at the detector. Show that you can express intensity I_n as a function of wavevector k_n. Derive the functional form of the fringe contrast (coherence function) as a function of the mirror position. **The intended function can be left in integral form.**

Hint: Start by solving the coherence function of two waves, then express it as a function of $k, \Delta x$. For a Gaussian spectrum, all frequency components have non-zero intensities. Therefore, an integral over all frequency components (or rather all wavenumbers) will yield the coherence function.

Let's start by solving I_{12}, again $T = \pi/\omega$:

$$For\ wave\ 1\text{: } \sin(\omega t - kx_1)$$

$$For\ wave\ 2\text{: } \sin(\omega t - kx_2 - \phi) = \sin(\omega t - kx_2 - k\Delta x)$$

$$I_{12} = \frac{2}{\mu_0 c}\langle E_1 E_2 \rangle = \frac{2}{\mu_0 c}\frac{\int_0^{\frac{\pi}{\omega}} E_1 E_2 \sin(\omega t - kx_1) * \sin(\omega t - kx_2 - k\Delta x)\, dt}{T}$$

$$= \frac{2}{\mu_0 c}\frac{\omega}{\pi}E_1 E_2 \int_0^{\frac{\pi}{\omega}} \sin(\omega t - kx_1) * \sin(\omega t - kx_2 - k\Delta x)\, dt$$

$$= \frac{1}{\mu_0 c}E_1 E_2 \cos(2k(x_2 - x_1)) = \frac{1}{\mu_0 c}E_1 E_2 \cos(2k\Delta x)$$

The total irradiance of two interfering waves is given by:

$$I_{tot} = I_1 + I_2 + I_{12} = I_1 + I_2 + \frac{1}{\mu_0 c}E_1 E_2 \cos(2k\Delta x)$$

$$I_{tot} = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos(2k\Delta x)$$

The integral over all wavenumbers produces the coherence function:

$$I(\Delta x) = \int_0^\infty [I_1(k) + I_2(k) + 2\sqrt{I_1(k) I_2(k)}\cos(2k\Delta x)]dk$$

Irradiance can be expressed as a function of $k = \frac{2\pi}{\lambda_0} = \frac{\omega}{c}$:

$$I_n = \langle E_n^2 \rangle = \frac{1}{2\mu_0 c} E_0^2 = \frac{f}{2\mu_0 \lambda} E_0^2 = \frac{\omega k}{2\mu_0} E_0^2 = \frac{c k^2}{2\mu_0} E_0^2$$