Photonics, Spring 2023

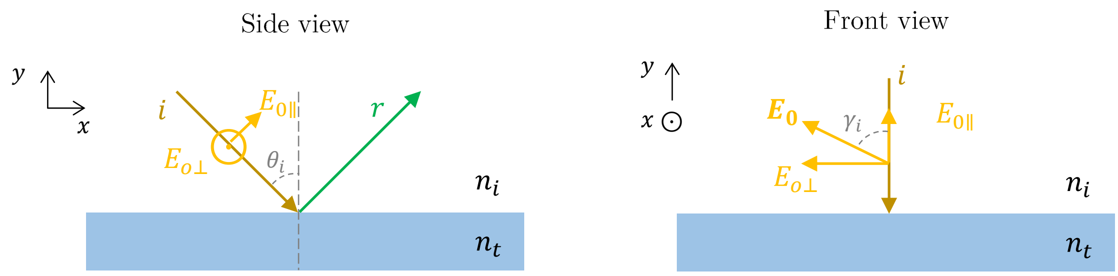
Exercises to be returned via Google Classroom by **14:00 next Tuesday 31.1.** (link sent by Joonas Mustonen via email). **Include your name and student number in the returned exercises**. For each problem, ½ a point will be awarded for an honest effort and 1 point for a well worked solution.

Exercise sessions are held on Tuesdays from 16:00 - 18:00 at Chemicum A121.

Exercise 2, 24.1.2023

1. Reflectance of polarized light (1 point)

A linearly polarized plane wave is incident on an interface between two linear, homogenous non-magnetic di-electric media. The angle between the plane of incidence (xy-plane) and direction of polarization is γ_i and the angle of incidence is θ_i. The reflectance components of the p- and s-polarized components are $R_\parallel$ and $R_\perp$, respectively. Write an expression for the total reflectance R.
Hint: start by writing $R = I_r/I_i$, where I_r is the total reflected intensity.



The wave amplitude can be separated to its parallel and perpendicular components. The parallel/perpendicular component amplitudes can be expressed as functions of the polarization angle:

$$E_{0i\perp} = E_{0i}\sin\gamma_i \qquad E_{0i\parallel} = E_{0i}\cos\gamma_i$$

Irradiance $I = \frac{c\epsilon}{2}E_0^2$ can be written for each component using the above equations:

$$I_{i\perp} = \frac{c\epsilon}{2}E_{0i\perp}^2 = \frac{c\epsilon}{2}E_{0i}^2\sin^2\gamma_i = I_i\sin^2\gamma_i \rightarrow I_i = \frac{I_{i\perp}}{\sin^2\gamma_i} \tag{1}$$

$$I_{i\parallel} = \frac{c\epsilon}{2}E_{0i\parallel}^2 = \frac{c\epsilon}{2}E_{0i}^2\cos^2\gamma_i = I_i\cos^2\gamma_i \rightarrow I_i = \frac{I_{i\parallel}}{\cos^2\gamma_i} \tag{2}$$

The total reflectance R can be expressed as the ratio of total reflected and incident irradiances:

$$R = \frac{I_r}{I_i} = \frac{I_{r\perp} + I_{r\parallel}}{I_i} = \frac{I_{r\perp}}{I_i} + \frac{I_{r\parallel}}{I_i} \qquad \| \quad Substitute\ Eqs. 1,2$$

$$R = \frac{I_{r\perp}}{I_{i\perp}}\sin^2\gamma_i + \frac{I_{r\parallel}}{I_{i\parallel}}\cos^2\gamma_i$$

$$R = R_\perp\sin^2\gamma_i + R_\parallel\cos^2\gamma_i$$

2. Reflectance of natural light (1 points)

Natural, or unpolarized light, is such that the angle γ_i of problem 1 changes rapidly and randomly. Derive an expression for the reflectance of natural light R_n in terms of $I_\parallel$ and $I_\perp$.

Hint: problem 1 should give the reflectance R as a function of γ_i. Since the polarization angle now varies randomly, take the time average of that result: $R_n = \langle R \rangle = \langle f(\gamma) R_\parallel + g(\gamma) R_\perp \rangle$.

We are now looking for the time average of total reflectance for unpolarized light.

$$R_n =< R(\gamma_i) > = < \cos^2 \gamma_i \, R_\parallel + \sin^2 \gamma_i \, R_\perp >$$

The polarization angle can be anywhere between $0 - 2\pi$, which will act as our integration limits:

$$R_n \;= \frac{\int_0^{2\pi} \cos^2 \gamma_i \, R_\parallel + \sin^2 \gamma_i \, R_\perp d\gamma}{\int_0^{2\pi} d\gamma}$$

$$R_n = \frac{1}{2\pi} \left(R_\parallel \int_0^{2\pi} \cos^2 \gamma_i \, d\gamma + R_\perp \int_0^{2\pi} \sin^2 \gamma_i \, R_\perp d\gamma \right)$$

$$R_n \;= \frac{1}{2\pi} \left(R_\parallel \pi + R_\perp \pi \right) = \frac{1}{2} \left(R_\parallel + R_\perp \right)$$

3. Reflection and transmission at a semiconductor-semiconductor interface (4 points)

A light wave with a free space wavelength of 890 nm that is propagating in GaAs becomes incident on AlGaAs. The refractive indices of GaAs and AlGaAs are 3.60 and 3.30, respectively.

a) Consider normal incidence. What are the reflection and transmission coefficients and the reflectance and transmittance?

At normal incidence the Fresnel equations are:

$$r_\parallel = \frac{n_t - n_i}{n_t + n_i} = \frac{3.30 - 3.60}{3.30 + 3.60} \approx -0.043 \qquad t_\parallel = \frac{2n_i}{n_t + n_i} = \frac{2 * 3.60}{3.30 + 3.60} \approx 1.043$$

$$r_\perp = \frac{3.60 - 3.30}{3.60 + 3.30} \approx 0.043 \qquad t_\perp = \frac{2n_i}{n_i + n_t} = \frac{2 * 3.60}{3.60 + 3.30} \approx 1.043$$

$$R = \left(\frac{n_2 - n_1}{n_2 + n_1}\right)^2 = r^2 = 0.043^2 \approx 0.0019 \qquad T = \frac{4n_1 n_2}{(n_2 + n_1)^2} \approx 0.9981$$

b) What is the Brewster angle (polarization angle θ_p) and the critical angle θ_c for total internal reflection for the wave incident on the GaAs/AlGaAs interface?

Brewster's angle is the angle of incidence at which the sum of incident and refracted angles $\theta_i + \theta_t = 90°$. This is the condition for total transmission of p-polarized light i.e., all reflected light is s-polarized. With Snell's law:

$$n_1 \sin\theta_i = n_1 \sin\theta_p = n_2 \sin\theta_t$$

$$n_1 \sin\theta_p = n_2 \sin\left(\frac{\pi}{2} - \theta_i\right)$$

$$n_1 \sin\theta_p = n_2 \cos\theta_p \quad \| \quad \sin\left(\frac{\pi}{2} - x\right) = \cos x$$

$$\frac{\sin\theta_p}{\cos\theta_p} = \tan\theta_p = \frac{n_2}{n_1}$$

$$\theta_p = \arctan\left(\frac{n_2}{n_1}\right) = \arctan\left(\frac{3.30}{3.60}\right) \approx 42.5°$$

The critical angle is the angle of incidence at which the refracted angle $\theta_t = 90°$. Again with Snell's law:

$$n_1 \sin\theta_i = n_1 \sin\theta_c = n_2 \sin\theta_t = n_2 \sin 90° = n_2$$

$$\theta_c = \operatorname{asin}\left(\frac{n_2}{n_1}\right) = \operatorname{asin}\left(\frac{3.30}{3.60}\right) \approx 66.4°$$

c) What is the reflection coefficient and the phase change in the reflected wave when the angle of incidence is $\theta_i = 79°$?

θ_t from Snell's law, or assuming $\theta_t = 90°$ with the argument that $\theta_i > \theta_c$ (produce same result):

$$n_1 \sin\theta_i = n_2 \sin\theta_t \rightarrow \theta_t = \operatorname{asin}\left(\frac{n_1}{n_2}\sin\theta_i\right) = \operatorname{asin}\left(\frac{3.6}{3.3}\sin 79°\right)$$

Fresnel's equations:

$$r_\parallel = \frac{n_t \cos\theta_i - n_i \cos\theta_t}{n_t \cos\theta_i + n_i \cos\theta_t} = \frac{3.3\cos 79° - 3.6\cos\left(\operatorname{asin}\left(\frac{3.6}{3.3}\sin 79°\right)\right)}{3.3\cos 79° + 3.6\cos\left(\operatorname{asin}\left(\frac{3.6}{3.3}\sin 79°\right)\right)} \approx -0.655 - 0.756i$$

$$r_\perp = \frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t} = \frac{3.6 \cos 79° - 3.3 \cos\left(\operatorname{asin}\left(\frac{3.6}{3.3}\sin 79°\right)\right)}{3.6 \cos 79° + 3.3 \cos\left(\operatorname{asin}\left(\frac{3.6}{3.3}\sin 79°\right)\right)} \approx -0.544 - 0.839i$$

$$R_\parallel = \left|r_\parallel\right|^2 = 1 \qquad R_\perp = |r_\perp|^2 = 1$$

The phase change under TIR for each component is given by:

$$\tan\left(\frac{\phi_\perp}{2}\right) = \frac{\sqrt{\sin^2\theta_i - n^2}}{\cos\theta_i} \quad \| \quad n = \frac{n_t}{n_i}$$

$$\phi_\perp \approx 123°$$

$$\tan\left(\frac{\phi_\parallel + \pi}{2}\right) = \frac{\sqrt{\sin^2\theta_i - n^2}}{n^2\cos\theta_i} \quad \| \quad n = \frac{n_t}{n_i}$$

$$\phi_\parallel \approx -49.08°$$

d) What is the penetration depth of the evanescent wave into the AlGaAs when $\theta_i = 79°$ and when $\theta_i = 89°$? Does the result make sense?

The penetration depth is the inverse of the attenuation coefficient α:

$$d = \frac{1}{\alpha} = \frac{\lambda_0}{2\pi n_t \sqrt{\left(\frac{n_i}{n_t}\right)^2 \sin^2\theta_i - 1}}$$

At $\theta_i = 79°$:

$$d = \frac{1}{\alpha} = \frac{890\text{nm}}{2\pi 3.3\sqrt{\left(\frac{3.6}{3.3}\right)^2 \sin^2 79° - 1}} \approx 112 \text{ nm}$$

At $\theta_i = 89°$:

$$d = \frac{1}{\alpha} = \frac{890\text{nm}}{2\pi 3.3\sqrt{\left(\frac{3.6}{3.3}\right)^2 \sin^2 89° - 1}} \approx 98.6 \text{ nm}$$

The penetration depth decreases as the angle approaches 90°.

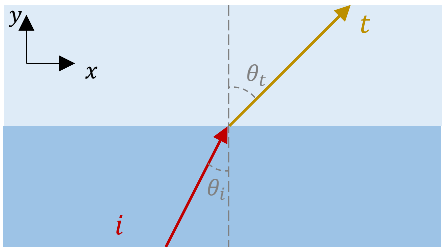
4. Evanescent wave (2 points)

Total internal reflection (TIR) of a plane wave from a boundary between a dense (n_1) and a denser (n_2) medium is accompanied by an evanescent wave propagating in medium 2. Derive the corresponding plane wave wavefunction and discuss how its magnitude varies with penetration into medium 2. The general wavefunction is:

$$\boldsymbol{E} = \boldsymbol{E_0} e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$$

Hint: draw a simple ray diagram of the general case of incident and transmitted rays (without TIR). Express the wave vector in terms of its orthogonal components (x, y). When $\theta_i > \theta_c$ the transmission angle becomes complex, where the imaginary part describes the Evanescent wave; the imaginary term allows for exponential decay of the resulting Evanescent wave. The goal is then to express both components of the wave vector as functions of θ_i, which remains real-valued. Using definitions of the refractive index $n = c/v = \lambda_0/\lambda$, the wavenumber $k = 2\pi/\lambda$ and Snell's law, strive to express the wave vector components as functions of the real-valued θ_i and readily known parameters n and λ_0.

The wave vector for the transmitted wave (where x and y lie on the plane of incidence) can be split into orthogonal components:



$$\boldsymbol{k_t} = k_{tx}\hat{x} + k_{ty}\hat{y}$$

$$k_{tx} = k_t \sin\theta_t$$

$$k_{ty} = k_t \cos\theta_t$$

We also want to express k in terms of more readily known parameters, namely n and λ_0:

$$n = \frac{c}{v} = \frac{\lambda_0}{\lambda} \rightarrow \lambda = \frac{\lambda_0}{n} \quad , \quad k = \frac{2\pi}{\lambda}$$

$$n = \frac{k\lambda_0}{2\pi} \quad , \quad k = \frac{2\pi n}{\lambda_0} \quad , \quad \frac{k_i}{k_t} = \frac{n_t}{n_i} \qquad 1,2,3$$

Then using Snell's law we can rewrite k_{ty}:

$$\frac{\sin\theta_i}{n_i} = \frac{\sin\theta_t}{n_t} \quad \| \quad Substitute\ Eq.1$$

$$\frac{k_i \sin\theta_i}{2\pi\lambda_0} = \frac{k_t \sin\theta_t}{2\pi\lambda_0}$$

$$k_i \sin\theta_i = k_t \sin\theta_t \quad \| \quad \sin x = \sqrt{1-\cos^2 x}$$

$$k_i \sin\theta_i = k_t\sqrt{1-\cos^2\theta_t}$$

$$k_t^2(1-\cos^2\theta_t) = \pm k_i^2 \sin^2\theta_i$$

$$k_t^2 \cos^2\theta_t = \pm(k_i^2 \sin^2\theta_i - k_t^2)$$

$$k_t \cos\theta_t = \pm\sqrt{k_i^2\sin^2\theta_i - k_t^2}$$

$$k_t \cos\theta_t = \pm k_t\sqrt{\frac{k_i^2}{k_t^2}\sin^2\theta_i - 1}$$

$$k_t \cos\theta_t = \pm\frac{2\pi n_t}{\lambda_0}\sqrt{\left(\frac{n_t}{n_i}\right)^2 \sin^2\theta_i - 1} \quad \| \quad Substitute\ Eqs. 2,3 \qquad 4$$

The term we have arrived at is actually the attenuation coefficient α. Now we can equate k_{ty} to Eq. 4:

$$k_{ty} = \pm i \frac{2\pi n_t}{\lambda_0} \sqrt{\left(\frac{n_t}{n_i}\right)^2 \sin^2\theta_i - 1} = \pm i\alpha \qquad 5$$

With Snell's law we can rewrite the sine term in k_{tx}, then substitute Eq. 2:

$$k_{tx} = k_t \sin\theta_t = k_t \frac{n_t}{n_i} \sin\theta_i = \frac{2\pi n_t^2}{\lambda_0 n_i} \sin\theta_i \qquad 6$$

The general form of the wavefunction is:

$$\boldsymbol{E} = \boldsymbol{E_0} e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} = \boldsymbol{E_{0t}} e^{i\left(k_x r_x + k_y r_y + k_z r_z - \omega t\right)}$$

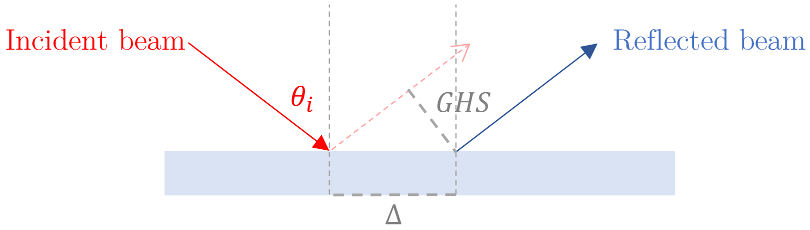
For our case omitting the z-components and substituting Eqs. 5, 6 we arrive at:

$$\boldsymbol{E_t} = \boldsymbol{E_{0t}} e^{i\left(x\frac{2\pi n_t^2}{\lambda_0 n_i}\sin\theta_i \pm i\alpha y - \omega t\right)} = \boldsymbol{E_{0t}} e^{\pm\alpha y} e^{i\left(x\frac{2\pi n_t^2}{\lambda_0 n_i}\sin\theta_i - \omega t\right)}$$

The first exponential term imposes an exponential decay of the wave amplitude.

5. **Goos-Hänchen shift** (2 point)

The Goos-Hänchen shift can be thought of as an addition to the law of reflection ($\theta_i = \theta_r$). Experiments conducted by Goos and Hänchen in the mid 1900's demonstrated that during total internal reflection, the reflection of a real beam with a finite width will appear to be displaced along the interface i.e., a spatial shift Δ occurs between the incident and reflected rays:



The Goos-Hänchen shift GHS is defined as the lateral shift with respect to reflected beam, and the displacement along the surface Δ is (http://www.scholarpedia.org/article/Goos-Hänchen_effect):

$$\Delta = -\frac{\lambda_0}{2\pi n_i \cos\theta_i}\frac{\partial\phi(\theta_i)}{\partial\theta_i}$$

where ϕ is the phase shift. A ray of light travelling in a glass medium ($n_1 = 1.460$) becomes incident on another glass medium ($n_2 = 1.430$). The free-space wavelength of the light ray is $\lambda_0 = 850$ nm and the angle of incidence is $\theta_i = 85°$. Estimate the Goos-Hänchen shift (GHS) in the reflected wave for the parallel and perpendicular field components. What can you conclude about the reflected waves, and would there be some applications for this effect?

We can solve the phase shifts form the familiar TIR case:

$$\phi_\parallel = -2\arctan\left(\frac{\sqrt{\sin^2\theta_i - n^2}}{n^2\cos\theta_i}\right) \quad \| \quad n = \frac{n_t}{n_i}$$

$$\phi_\perp = -2\arctan\left(\frac{\sqrt{\sin^2\theta_i - n^2}}{\cos\theta_i}\right)$$

Calculating the derivatives (https://www.derivative-calculator.net):

$$\frac{d\phi_\parallel}{d\theta} = \frac{2n^2\sin\theta_i}{\left((n^2+1)\cos^2\theta_i - 1\right)\sqrt{\sin^2\theta_i - n^2}}$$

$$\frac{d\phi_\perp}{d\theta} = -\frac{2\sin\theta_i}{\sqrt{\sin^2\theta_i - n^2}}$$

The GHS can now be solved:

$$\cos\theta_i = \frac{\text{GHS}}{\Delta} \rightarrow GHS = \Delta\cos\theta_i = -\cancel{\cos\theta_i}\frac{\lambda_0}{2\pi n_i\cancel{\cos\theta_i}}\frac{\partial\phi(\theta_i)}{\partial\theta_i}$$

$$GHS = -\frac{\lambda_0}{2\pi n_i}\frac{\partial\phi(\theta_i)}{\partial\theta_i} = -\frac{\lambda_0}{\cancel{2}\pi n_i}\frac{\cancel{2}n^2\sin\theta_i}{\left((n^2+1)\cos^2\theta_i - 1\right)\sqrt{\sin^2\theta_i - n^2}}$$

$$GHS_\parallel = -\frac{850\text{ nm}}{\pi * 1.46} * \frac{\left(\frac{1.43}{1.46}\right)^2\sin 85°}{\left(\left(\left(\frac{1.43}{1.46}\right)^2+1\right)\cos^2 85° - 1\right)\sqrt{\sin^2 85° - \left(\frac{1.43}{1.46}\right)^2}} \approx 988.5\text{ nm}$$

$$GHS_\perp = -\frac{\lambda_0}{2\pi n_i}\frac{\partial\phi_\perp(\theta_i)}{\partial\theta_i} = -\frac{\lambda_0}{2\pi n_i}\frac{2\sin\theta_i}{\sqrt{\sin^2\theta_i - n^2}}$$

$$GHS_\perp = \frac{850\text{ nm}}{2\pi 1.46} * \frac{2\sin 85°}{\sqrt{\sin^2 85° - \left(\frac{1.43}{1.46}\right)^2}} \approx 1015\text{ nm}$$

Different polarizations are shifted by different amounts, so separation of polarization modes is possible. Also, the phase change is different for the two reflections. Due to these differences, which are dependent on the ratio of refractive indices, the GHS has potential for probing of (e.g. biological) materials.