Photonics, Spring 2023

Exercise 1, 17.1.2023 – MODEL SOLUTIONS

1. Maxwell's equations and waves (1 point)

Derive the free space wave equation for electric and magnetic fields from Maxwell's equations.

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} \tag{1}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{2}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{3}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \left(\boldsymbol{J} + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right) \tag{4}$$

General wave equation form:

$$\frac{\partial^2 \boldsymbol{u}}{\partial^2 t} = c^2 \nabla^2 \boldsymbol{u}$$

No charge or current in vacuum $\boldsymbol{\rho = 0, J = 0}$. Solve for electric field wave equation by taking the curl of Eq. 3, using identity $\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$:

$$\nabla \times (\nabla \times \boldsymbol{E}) = \nabla \times \left(-\frac{\partial \boldsymbol{B}}{\partial t} \right)$$

$$\nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} = -\frac{\partial}{\partial \mathrm{t}} (\nabla \times \boldsymbol{B}) \quad \| \quad \nabla \cdot \boldsymbol{E} = \boldsymbol{0}, \text{ substituting Eq. 4}$$

$$-\nabla^2 \boldsymbol{E} = -\frac{\partial}{\partial t} \left(\mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right)$$

$$\nabla^2 \boldsymbol{E} = \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \boldsymbol{E}$$

Similar approach for magnetic field, taking curl of Eq. 4, using same identity as before.

$$\nabla \times (\nabla \times \boldsymbol{B}) = \nabla \times \left(\mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right)$$

$$\nabla(\nabla \cdot \boldsymbol{B}) - \nabla^2 \boldsymbol{B} = -\frac{\partial}{\partial t} (\nabla \times \boldsymbol{E}) \quad \| \quad \nabla \cdot \boldsymbol{B} = \boldsymbol{0}, \text{ substituting Eq. 3}$$

$$\nabla^2 \boldsymbol{B} = \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \boldsymbol{B}$$

2. **Reflectance and transmittance** (3 points)

Starting from the Fresnel equations:

$$r_\parallel = \left(\frac{E_{0r}}{E_{0i}}\right)_\parallel = \frac{n_t\cos\theta_i - n_i\cos\theta_t}{n_t\cos\theta_i + n_i\cos\theta_t}$$

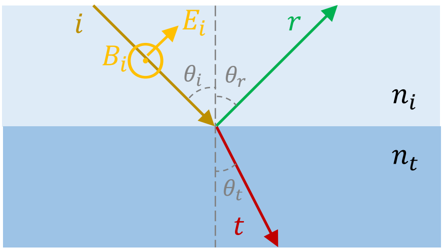
$$t_\parallel = \left(\frac{E_{0t}}{E_{0i}}\right)_\parallel = \frac{2n_i\cos\theta_i}{n_t\cos\theta_i + n_i\cos\theta_t}$$

$$r_\perp = \left(\frac{E_{0r}}{E_{0i}}\right)_\perp = \frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t}$$

$$t_\perp = \left(\frac{E_{0t}}{E_{0i}}\right)_\perp = \frac{2n_i\cos\theta_i}{n_i\cos\theta_i + n_t\cos\theta_t}$$

a) Derive the expressions for the amplitude coefficients for normal incidence for both transverse magnetic (TM) and electric (TE) waves.

Wave can be split into perpendicular components: $\boldsymbol{E}$ parallel/perpendicular, $\boldsymbol{B}$ perpendicular/parallel to plane of incidence. At normal incidence $\theta_i = \theta_r = \theta_t = 0 \rightarrow \cos\theta = 1$:



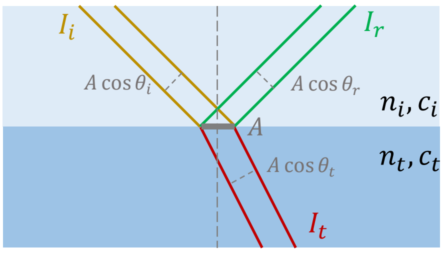
$$r_\parallel = \left(\frac{E_{0r}}{E_{0i}}\right)_\parallel = \frac{n_t - n_i}{n_t + n_i} \quad , \quad t_\parallel = \left(\frac{E_{0t}}{E_{0i}}\right)_\parallel = \frac{2n_i}{n_t + n_i}$$

$$r_\perp = \left(\frac{E_{0r}}{E_{0i}}\right)_\perp = \frac{n_i - n_t}{n_i + n_t} \quad , \quad t_\perp = \left(\frac{E_{0t}}{E_{0i}}\right)_\perp = \frac{2n_i}{n_i + n_t}$$

b) Derive the expression for coefficients of transmittance and reflectance at normal incidence:

$$R = \left(\frac{n_2 - n_1}{n_2 + n_1}\right)^2 \qquad T = \frac{4n_1 n_2}{(n_2 + n_1)^2}$$

The coefficients are ratios of reflected/transmitted powers. 'Light power' i.e., irradiance $I = \frac{c\varepsilon}{2}E_0^2$ (Hecht, W/m²), at normal incidence $\theta_i = \theta_r = \theta_t = 0 \rightarrow \cos\theta = 1$.



$$R = \frac{I_r \cancel{A\cos\theta_r}}{I_i \cancel{A\cos\theta_i}} = \frac{\frac{\cancel{c_r\varepsilon_r}}{\cancel{2}}E_{0r}^2}{\frac{\cancel{c_i\varepsilon_i}}{\cancel{2}}E_{0_i}^2} = \left(\frac{E_{0r}}{E_{0i}}\right)^2 = r^2 = \left(\frac{n_i - n_t}{n_i + n_t}\right)^2 \quad \parallel \quad c_r = c_i \ , \ \varepsilon_i = \varepsilon_r \ , \ r_\parallel^2 = r_\perp^2$$

$$T = \frac{I_t \cancel{A\cos\theta_t}}{I_i \cancel{A\cos\theta_i}} = \frac{\frac{c_t\varepsilon_t}{\cancel{2}}E_{0t}^2}{\frac{c_i\varepsilon_i}{\cancel{2}}E_{0_i}^2} = \frac{\frac{\cancel{c}}{n_t}\varepsilon_{rt}\cancel{\epsilon_0}E_{0t}^2}{\frac{\cancel{c}}{n_i}\varepsilon_{ri}\cancel{\epsilon_0}E_{0i}^2} \quad \parallel \quad c_t = \frac{c}{n_t}$$

Refractive index: $n = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \rightarrow \varepsilon_r = n^2\frac{\mu_0}{\mu}$, assuming $\mu_i = \mu_t = \mu_0 \rightarrow \varepsilon_r = n^2$:

$$T = \frac{n_t\varepsilon_0 E_{0t}^2}{n_i\varepsilon_0 E_{0i}^2} = \frac{n_t}{n_i}t^2 = \frac{n_t}{n_i}\left(\frac{2n_i}{n_i + n_t}\right)^2 = \frac{4n_i^{\cancel{2}}n_t}{\cancel{n_i}(n_i + n_t)^2} = \frac{4n_i n_t}{(n_i + n_t)^2} \qquad \parallel \quad t_\parallel^2 = t_\perp^2$$

c) What is the percentage of reflected irradiance at an air-glass interface (normal incidence, $n_{air} = 1, n_{glass} = 1.5$)?

$$n_i = 1, n_t = 1.5 \rightarrow R = \left(\frac{n_i - n_t}{n_i + n_t}\right)^2 = 0.04 = 4\%$$

3. **Total internal reflection** (3 points)

A ray of light is incident at an interface between two linear dielectric media ($n_1 = 1.45, n_2 = 1.33$, vacuum wavelength $\lambda = 1.064\ \mu\text{m}$)

a) Calculate the minimum angle for total internal reflection

During total internal reflection $\theta_t = 90° \rightarrow \sin\theta_t = 1$. Using Snell's law:

$$n_i \sin\theta_c = n_t \sin\theta_t = n_t$$

$$\theta_c = \sin^{-1}\left(\frac{n_t}{n_i}\right) = \sin^{-1}\left(\frac{1.33}{1.45}\right) \approx 66.5°$$

b) Calculate the phase change in reflection, when $\theta_i = 70°$

$\theta_i > \theta_c \rightarrow$ total internal reflection (and phase shift) occurs.

Phase shift for s-polarization in upon TIR (found from slides):

$$\tan\left(\frac{\phi_\perp}{2}\right) = \frac{\sqrt{\sin^2\theta_i - n^2}}{\cos\theta_i} \quad \| \quad n = \frac{n_2}{n_1}$$

$$\phi_\perp = 2\tan^{-1}\left(\frac{\sqrt{\sin^2\theta_i - n^2}}{\cos\theta_i}\right)$$

$$\phi_\perp = 2\tan^{-1}\left(\frac{\sqrt{\sin^2 70° - (1.33/1.45)^2}}{\cos 70°}\right) \approx 61.7°$$

Phase shift for p-polarization in upon TIR (found from slides):

$$\tan\left(\frac{\phi_\parallel + \pi}{2}\right) = \frac{\sqrt{\sin^2\theta_i - n^2}}{n^2\cos\theta_i} \quad \| \quad n = \frac{n_2}{n_1}$$

$$\phi_\parallel = 2\tan^{-1}\left(\frac{\sqrt{\sin^2\theta_i - n^2}}{n^2\cos\theta_i}\right) - \pi$$

$$\phi_\parallel = 2\tan^{-1}\left(\frac{\sqrt{\sin^2 70° - (1.33/1.45)^2}}{(1.33/1.45)^2\cos 70°}\right) - 180° \approx -109°$$

c) Calculate the penetration depth, when $\theta_i = 70°$

When $\theta_i > \theta_c$, boundary conditions (Maxwellian analysis at interface) dictate that there must be an electric field in the reflecting medium. A wave therefore still transmits into the medium, traveling along the surface and attenuating into the medium → Evanescent wave. The transmitted (Evanescent) wave vector becomes complex:

$$\boldsymbol{k} = k_{tx}\hat{x} + k_{ty}\hat{y} = k_{tx}\hat{x} + i\alpha\hat{y}$$

The transmitted wave becomes:

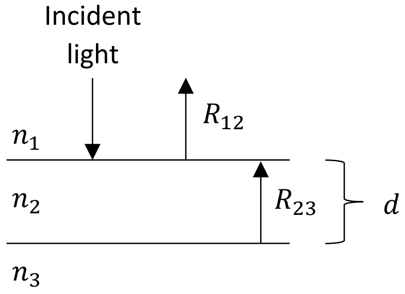
$$\boldsymbol{E}_{t\perp} \propto e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})} = e^{i(\omega t - (k_{tx}\hat{x} + i\alpha\hat{y}))} = e^{-\alpha y}e^{-i(\omega t - k_t\hat{x})}$$

Where $\alpha = \frac{2\pi n_2}{\lambda}\sqrt{\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_i - 1}$ (Hecht) is the attenuation coefficient. The penetration depth is its inverse:

$$d = \frac{1}{\alpha} = \frac{\lambda}{2\pi n_2\sqrt{\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_i - 1}} \approx 0.57\mu\text{m}$$

4. **Antireflection coatings** (2 points)

a) Consider three dielectric media, with flat and parallel boundaries with refractive indices n_1, n_2 and n_3. Show that for normal incidence the reflection coefficient between layer 1 and 2 is the same as between layers 2 and 3 if $n_2 = \sqrt{n_1 n_3}$



At normal incidence:

$$R_{12} = \left(\frac{n_2 - n_1}{n_1 + n_2}\right)^2 \quad , \quad R_{23} = \left(\frac{n_3 - n_2}{n_2 + n_3}\right)^2$$

$$n_2 = \sqrt{n_1 n_3} \;\rightarrow\; n_1 = \frac{n_2^2}{n_3}$$

$$\sqrt{R_{12}} = \frac{n_2 - n_1}{n_1 + n_2} = \frac{n_2 - \frac{n_2^2}{n_3}}{\frac{n_2^2}{n_3} + n_2} = \frac{n_2\left(1 - \frac{n_2}{n_3}\right)}{n_2\left(1 + \frac{n_2}{n_3}\right)} = \frac{1 - \frac{n_2}{n_3}}{1 + \frac{n_2}{n_3}}$$

$$= \frac{\frac{1}{n_3}(n_3 - n_2)}{\frac{1}{n_3}(n_3 + n_2)} = \frac{n_3 - n_2}{n_3 + n_2} = \sqrt{R_{23}}$$

b) What should be the refractive index and thickness of an antireflection coating designed to operate at $\lambda_0 = 1.064\ \mu\mathrm{m}$ on a BK-7 lens in air?

The first medium is air, second the AR coating (refractive index from internet), and the third the BK-7 lens:

$$n_1 = 1 \quad , \quad n_3(\lambda = 1.064\ \mu\mathrm{m}) \approx 1.507$$

For the two reflected waves to cancel out $R_{12} = R_{23}$, as in part a):

$$n_2 = \sqrt{n_1 n_2} \approx 1.228$$

For the reflected waves to cancel out, they also need to be in opposite phases. This requires that the thickness of the coating must correspond to odd multiples of the quarter wavelength of the coating (two-way propagation in the coating corresponding to half wavelength):

$$d = m\frac{\lambda_3}{4} \qquad \| \qquad \lambda = \frac{\lambda_0}{n} \quad , \quad m \in \text{odd itegers}$$

$$d = m\frac{\lambda_0}{4n_2} \approx 0.22\ \mu\mathrm{m} \cdot m$$

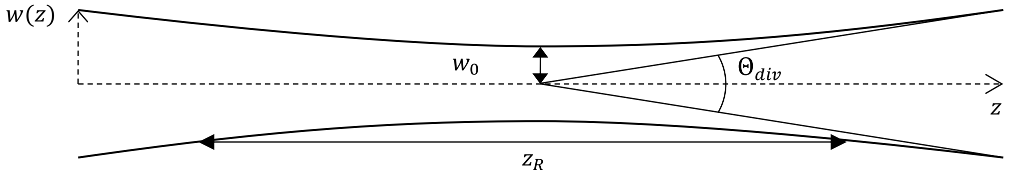
Alternatively, following lecture notes: AR coating should generate phase difference of $m\pi$ ($m \in$ odd itegers) between the two reflected waves for them to cancel out. The phase difference per unit length in the coating is equivalent to the propagation constant:

$$k_2 = \frac{2\pi}{\lambda_2} = \frac{2\pi n_2}{\lambda_0} \quad \rightarrow \quad m\pi = k_2 2d = \frac{2\pi n_2}{\lambda_0} 2d$$

$$d = \frac{m\lambda_0}{4n_2} \approx 0.22\ \mu\mathrm{m} \cdot m, \quad m \in \text{odd itegers}$$

5. **Gaussian beam** (1 point)

Estimate the divergence Θ and Rayleigh range z_R of a Gaussian beam from a HeNe laser with beam width $2w_0 = 1$ mm at $z = 0$. After traversing 10 m through vacuum, what will the beam width be?



Beam divergence is the angular expansion of the beam in the far field (equation from course material or e.g. Wikipedia):

$$\Theta = \frac{2\lambda}{\pi w_0} \approx 0.8 \text{ mrad} = 0.046° \quad \| \quad \lambda = 632.8 \text{ nm}$$

Rayleigh range is the distance along the beam propagation direction, from the waist z_0, at which the cross-sectional area of the beam has doubled:

$$z_R = \frac{\pi w_0^2}{\lambda} \approx 1.24 \text{ m}$$

Beam diameter at 10 m:

$$2w(z) = 2w_0 \sqrt{1 + \left(\frac{z}{z_R}\right)^2} \approx 8.1 \text{ mm}$$