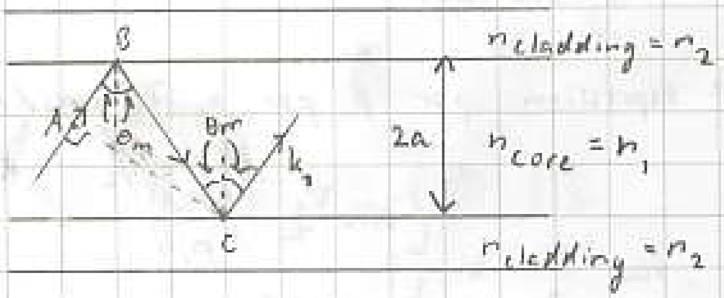
Photonics 17 Risto Montonen

1) Waveguide condition

Slab waveguide



Light traveling from A to C experiences two total internal replections (TIR). This causes a phase sift of $2\phi_m$, where $\phi_m = \phi(\theta_m)$ indicating that ϕ depends on incident angle θ_m.

Propagation corresponds to a phase sift of $(AB+BC)k_1$, where $k_1 = kn_1 = \frac{2\pi n_1}{\lambda}$. k and λ in vacuum.

Only waves with constructive interference can propagate in the slab. Therefore the phase difference between A and C must be a multiple of 2π.

$$\Rightarrow \quad \Delta\phi = (AB+BC)k_1 - 2\phi_m = 2\pi m, \quad m = 0,1,2,\dots$$

$$\cos\theta_m = \frac{2a}{BC} \Rightarrow BC = \frac{2a}{\cos\theta_m}$$

$$\cos 2\theta_m = \frac{AB}{BC} \Rightarrow AB = BC\cos 2\theta_m = \frac{2a\cos 2\theta_m}{\cos\theta_m}$$

$$\Rightarrow AB + BC = \frac{2a\cos 2\theta_m}{\cos\theta_m} + \frac{2a}{\cos\theta_m} = \frac{2a}{\cos\theta_m}\left(\cos 2\theta_m + 1\right)$$

$$\boxed{\cos(2x) = 2\cos^2 x - 1}$$

$$= \frac{2a}{\cancel{\cos\theta_m}}\left[2\cos^2\theta_m \cancel{-1+1}\right] = 4a\cos\theta_m$$

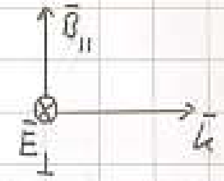
$$\Rightarrow \Delta\phi = 4a\cos\theta_m \cdot \frac{\cancel{2}\pi n_1}{\lambda} - \cancel{2}\phi_m = \cancel{2}m\pi$$

$$\frac{4\pi a n_1}{\lambda}\cos\theta_m - \phi_m = m\pi \quad \square$$

2) Let's consider planar dielectric waveguide with a core thickness of $2a = 20\mu m$, $n_{core} = n_1 = 1{,}455$, $n_{cladding} = n_2 = 1{,}440$, and $\lambda = 900\ nm$.

The expression for ϕ for a TE mode is:

$$\tan\frac{\phi_m}{2} = \frac{\left(\sin^2\theta_m - \left(\frac{n_2}{n_1}\right)^2\right)^{1/2}}{\cos\theta_m}.$$

TE: 

In problem 1 we got:

$$\frac{4\pi a n_1}{\lambda}\cos\theta_m - \phi_m = m\pi \Rightarrow \phi_m = \frac{4\pi a n_1}{\lambda}\cos\theta_m - m\pi.$$

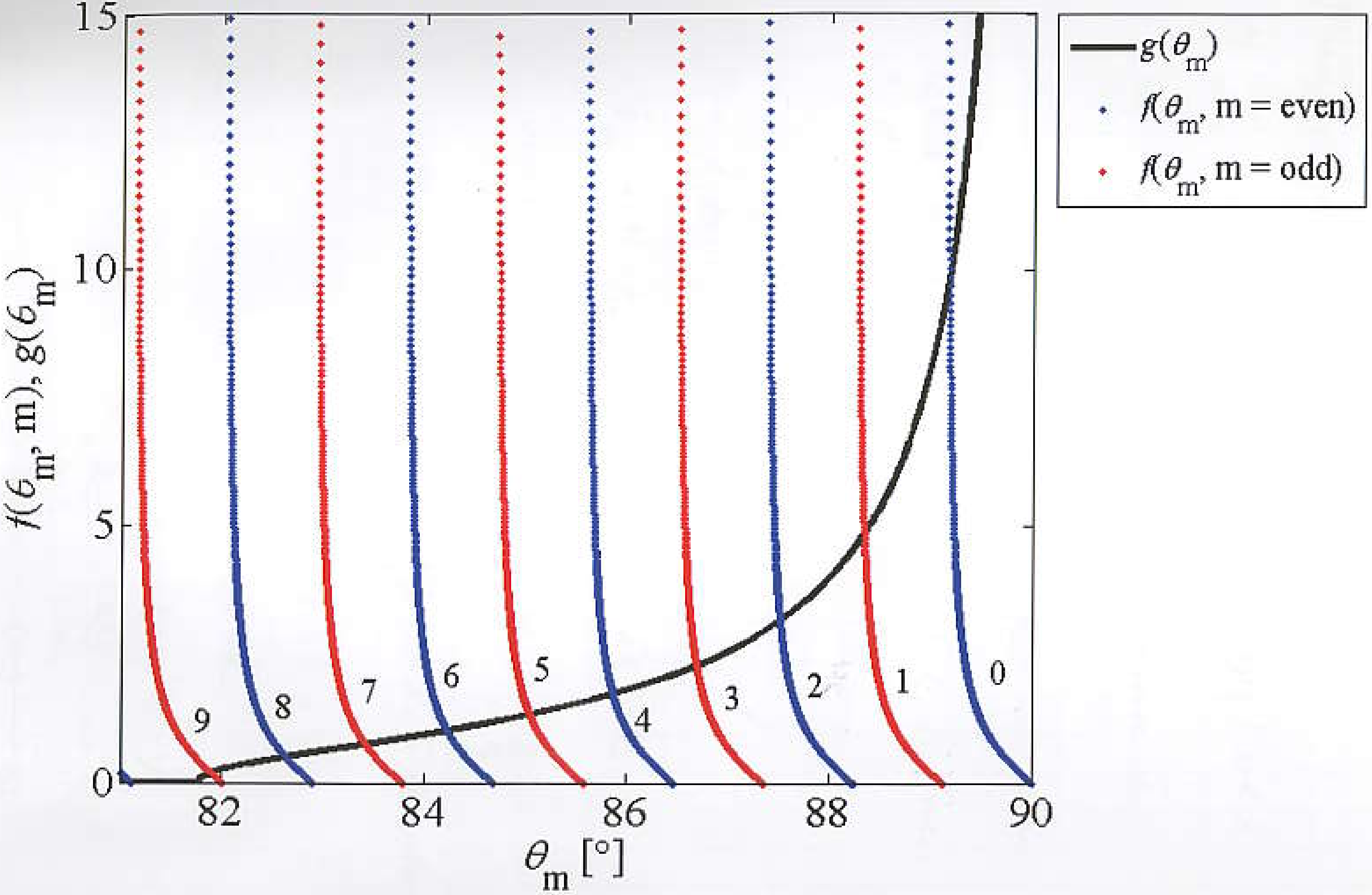
Now we get

$$\underbrace{\tan\left(\frac{2\pi a n_1}{\lambda}\cos\theta_m - \frac{m\pi}{2}\right)}_{=f(\theta_m, m)} = \underbrace{\frac{\left(\sin^2\theta_m - \left(\frac{n_2}{n_1}\right)^2\right)^{1/2}}{\cos\theta_m}}_{=g(\theta_m)}.$$

Both sides are functions of θ_m. Plotting $f(\theta_m, m)$ and $g(\theta_m)$ into a same graph, solutions for θ_m are obtained from intersections, see Fig. 1.

Solution for $m = 0 \ldots 9$ is:

m	θ_m
0	$89{,}2°$
1	$88{,}3°$
2	$87{,}5°$
3	$86{,}7°$
4	$85{,}9°$
5	$85{,}0°$
6	$84{,}2°$
7	$83{,}4°$
8	$82{,}6°$
9	$81{,}9°$



15
10
5
0
82
84
86
88
90
θ_m [°]
$f(\theta_m, m), g(\theta_m)$
$g(\theta_m)$
$f(\theta_m, m = \text{even})$
$f(\theta_m, m = \text{odd})$
0
1
2
3
4
5
6
7
8
9

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3) Waveguide mode shapes

a) What is the phase difference between rays 1 and 2 at C?

Ray 1 propagating from A to C experiences a phase sift of

$$k_1 AC - \phi_m = \frac{2\pi n_1}{\lambda} AC - \phi_m .$$

Instead, ray 2 propagating from A' to C experiences a phase sift of

$$k_1 A'C = \frac{2\pi n_1}{\lambda} A'C.$$

Phase difference:

$$\Phi_m = k_1 AC - \phi_m - k_1 A'C = k_1 (AC - A'C) - \phi_m .$$

Geometry: $\cos\theta_m = \frac{a-y}{AC} \Rightarrow AC = \frac{a-y}{\cos\theta_m}$

$$\cos(\pi - 2\theta_m) = \frac{A'C}{AC} \Rightarrow A'C = AC\cos(\pi - 2\theta_m)$$

$$\Rightarrow A'C = \frac{a-y}{\cos\theta_m}\cos(\pi - 2\theta_m)$$

$$= -\frac{a-y}{\cos\theta_m}\cos 2\theta_m$$

$\boxed{\cos 2x = 2\cos^2 x - 1}$

$$= -\frac{a-y}{\cos\theta_m}(2\cos^2\theta_m - 1)$$

$$= \frac{a-y}{\cos\theta_m} - \frac{a-y}{\cancel{\cos\theta_m}} 2\cos^{\cancel{2}}\theta_m$$

$$= \frac{a-y}{\cos\theta_m} - 2(a-y)\cos\theta_m$$

$$\Rightarrow \quad AC - A'C = \cancel{\frac{a-y}{\cos\theta_m}} - \cancel{\frac{a-y}{\cos\theta_m}} + 2(a-y)\cos\theta_m$$

$$= 2(a-y)\cos\theta_m$$

$$\Rightarrow \Phi_m = k_1(AC - A'C) - \phi_m = 2k_1(a-y)\cos\theta_m - \phi_m$$

In problem 1 we got

$$\frac{4\pi a n}{\lambda}\cos\theta_m - \phi_m = 2k_1 a\cos\theta_m - \phi_m = m\pi$$

$$\Rightarrow \cos\theta_m = \frac{m\pi + \phi_m}{2k_1 a}$$

Substituting this into Φ_m we get

$$\Phi_m = \Phi_m(y) = \cancel{2k_1}(a-y)\frac{m\pi + \phi_m}{\cancel{2k_1}a} - \phi_m$$

$$= \frac{\cancel{a}(m\pi + \phi_m)}{\cancel{a}} - \frac{y(m\pi + \phi_m)}{a} - \phi_m$$

$$= m\pi + \cancel{\phi_m} - \frac{y}{a}(m\pi + \phi_m) - \cancel{\phi_m}$$

$$= m\pi - \frac{y}{a}(m\pi + \phi_m) \quad \square$$

b) Field variation as a function of y.

Interfering waves at C:

$$E(y) = A\cos(\omega t) + A\cos[\omega t + \Phi_m(y)]$$
$$= A\left[\sin\left(\frac{\pi}{2} - \omega t\right) + \sin\left[\frac{\pi}{2} - (\omega t + \Phi_m(y))\right]\right]$$

$$\sin A + \sin B = 2\sin\left[\frac{1}{2}(A+B)\right]\cos\left[\frac{1}{2}(A-B)\right]$$

$$= 2A\sin\left[\frac{1}{2}\left(\frac{\pi}{2} - \omega t + \frac{\pi}{2} - (\omega t + \Phi_m(y))\right)\right]$$
$$\cdot \cos\left[\frac{1}{2}\left(\frac{\pi}{2} - \omega t - \frac{\pi}{2} + (\omega t + \Phi_m(y))\right)\right]$$

$$E(y) = 2A\sin\left[\tfrac{1}{2}\left(\pi - 2\omega t - \Phi_m(y)\right)\right]\cos\left[\tfrac{1}{2}\Phi_m(y)\right]$$

$$= 2A\sin\left[\tfrac{\pi}{2} - \left(\omega t + \tfrac{1}{2}\Phi_m(y)\right)\right]\cos\left[\tfrac{1}{2}\Phi_m(y)\right]$$

$$\cos A = \sin\left(\tfrac{\pi}{2} - A\right)$$

$$= 2A\cos\left[\omega t + \tfrac{1}{2}\Phi_m(y)\right]\cos\left[\tfrac{1}{2}\Phi_m(y)\right]$$

This is form of $E_0\cos[\omega t + \alpha]$, where

$$E_0 = 2A\cos\left[\tfrac{1}{2}\Phi_m(y)\right] = 2A\cos\left[\frac{m\pi}{2} - \frac{y}{2a}(m\pi + \phi_m)\right]$$

and $\alpha = \frac{1}{2}\Phi_m(y)$.

c) Let's set $A = 1$.

To plot E_0 we need to find ϕ_m for $m = 0, 1, 2$, the first three modes.

Considering problem 2 with parameters $a = 10\,\mu m$, $n_1 = 1{,}455$, $n_2 = 1{,}440$ and $\lambda = 1{,}3\,\mu m$ results into following θ_m and ϕ_m angles for TE modes

m	θ_m	ϕ_m
0	88,8°	163,8°
1	87,7°	147,0°
2	86,5°	129,7°