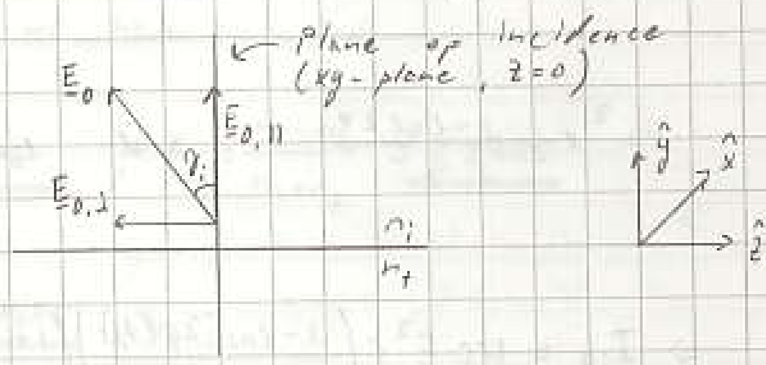
1) "Side view"

"Front view"



Angle between polarization direction and plane of incidence is γ

$$E_{0,\perp,i} = E_0 \sin\gamma_i$$
$$E_{0,\parallel,i} = E_0 \cos\gamma_i$$

Irradiance = time average of the magnitude of the Poynting vector, $\underline{S}$.

$$I = \langle |\underline{S}| \rangle = v\varepsilon \langle |\underline{E}_0|^2 \rangle = \frac{v\varepsilon}{2} E_0^2$$

Linearly polarized light $\Rightarrow$ γ_i does not depend on time

$$\Rightarrow \quad I_{i\perp} = I_i \sin^2\gamma_i \quad \text{and} \quad I_{i\parallel} = I_i \cos^2\gamma_i$$
$$I_i = \frac{I_{i\perp}}{\sin^2\gamma_i} \qquad I_i = \frac{I_{i\parallel}}{\cos^2\gamma_i}$$

Total reflectance:

$$R = \frac{I_r}{I_i} = \frac{I_{r\perp} + I_{r\parallel}}{I_i} = \overbrace{\frac{I_{r\perp}}{I_{i\perp}}}^{R_\perp} \sin^2\gamma_i + \overbrace{\frac{I_{r\parallel}}{I_{i\parallel}}}^{R_\parallel} \cos^2\gamma_i$$

$$= \underline{\underline{R_\perp \sin^2\gamma_i + R_\parallel \cos^2\gamma_i}}$$

2) When γ_i changes rapidly and randomly we need to calculate the time average of that as well.

$$E_{i\parallel} = E_0 \sin\left(\underline{k}\cdot\underline{z} - \omega t\right)\cos[\gamma_i(t)] = E_0 \sin[\varphi(t)]\cos[\gamma_i(t)]$$

$$\Rightarrow I_{i\parallel} = vc\langle E_{i\parallel}^2\rangle = vc\,E_0^2\langle \sin^2[\varphi(t)]\cos^2[\gamma_i(t)]\rangle$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

$$\Rightarrow I_{i\parallel} = vc\,E_0^2\left\langle\left(\frac{1-\cos[2\varphi(t)]}{2}\right)\left(\frac{1+\cos[2\gamma_i(t)]}{2}\right)\right\rangle$$

$$= \underbrace{vc\,E_0^2\,\frac{1}{2}}_{I_i}\,\frac{1}{2}\left\langle 1+\cancel{\cos[2\gamma_i(t)]}^{\to 0} - \cancel{\cos[2\varphi(t)]}^{\to 0} - \cancel{\cos[2\varphi(t)]\cos[2\gamma_i(t)]}^{\to 0}\right\rangle$$

$$= \frac{I_i}{2} \quad \Rightarrow \quad I_i = 2I_{i\parallel}$$

Similarly $E_{i\perp} = E_0 \sin[\varphi(t)]\sin[\gamma_i(t)]$

$$\Rightarrow I_{i\perp} = vc\langle E_{i\perp}^2\rangle = \frac{I_i}{2} \quad \Leftrightarrow I_i = 2I_{i\perp}$$

Total reflectivity:

$$R_n = \frac{I_{r\parallel}}{I_i} + \frac{I_{r\perp}}{I_i} = \frac{I_{r\parallel}}{2I_{i\parallel}} + \frac{I_{r\perp}}{2I_{i\perp}} = \underline{\underline{\frac{1}{2}\left(R_\parallel + R_\perp\right)}}$$

Time average:

$$\langle f(t)\rangle = \frac{1}{\Delta T}\int_t^{t+\Delta T} f(t')\,dt'$$

Photonics 17 Risto Montonen

3) A light wave with free space wavelength of 890 nm that is propagating in GaAs becomes incident on AlGaAs.

$\lambda_0 = 890$ nm , $n_{AlGaAs} = n_t = 3{,}30$, $n_{GaAs} = n_i = 3{,}60$

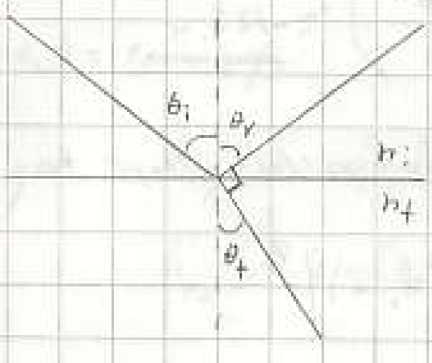
a) At normal incidence we have

$$r_{\parallel} = \frac{n_t - n_i}{n_i + n_t} \approx \underline{\underline{-0{,}0435}} \quad , \quad r_{\perp} = \frac{n_i - n_t}{n_i + n_t} \approx \underline{\underline{0{,}0435}}$$

$$t_{\parallel} = t_{\perp} = \frac{2n_i}{n_i + n_t} \approx \underline{\underline{1{,}0435}}$$

$$R = |r|^2 \approx \underline{\underline{0{,}0019}} \quad , \quad T = \frac{4 n_i n_t}{(n_i + n_t)^2} \approx \underline{\underline{0{,}9981}}$$

b) Brewster angle is found when the reflected and transmitted wave are perpendicular to each other. At that angle $r_{\parallel}$ goes to zero.



$\theta_r = \theta_i$

$\theta_r + \theta_t = 90°$

$$n_i \sin\theta_i = n_t \sin\theta_t$$
$$n_i \sin\theta_i = n_t \sin(90° - \theta_r)$$
$$n_i \sin\theta_i = n_t \cos\theta_i$$
$$\frac{\sin\theta_i}{\cos\theta_i} = \frac{n_t}{n_i}$$
$$\tan\theta_i = \frac{n_t}{n_i}$$

$$\theta_i = \tan^{-1}\frac{n_t}{n_i} \approx \underline{\underline{42{,}5° = \theta_B}}$$

Critical angle: $\theta_c = \sin^{-1}\frac{n_t}{n_i} \approx \underline{\underline{66,4^\circ}}$

c) $\theta_i = 77^\circ$ $\qquad n_i \sin\theta_i = n_t \sin\theta_t$

$$\theta_t = \sin^{-1}\left(\frac{n_i}{n_t}\sin\theta_i\right)$$

$$r_{\parallel} = \frac{n_t\cos\theta_i - n_i\cos\theta_t}{n_i\cos\theta_t + n_t\cos\theta_i} \approx \underline{\underline{-0,6550 - 0,7556\,i}}$$

$$r_{\perp} = \frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t} \approx \underline{\underline{-0,5441 - 0,8390\,i}}$$

$$R_{\parallel} = |r_{\parallel}|^2 = \underline{\underline{1}} \qquad R_{\perp} = |r_{\perp}|^2 = \underline{\underline{1}}$$

$$\varphi_{\parallel} = -2\tan^{-1}\left(\frac{\sqrt{\sin^2\theta_i - n^2}}{n^2\cos\theta_i}\right) + \pi \qquad n = \frac{n_t}{n_i}$$

$$\approx \underline{\underline{49,1^\circ}}$$

$$\varphi_{\perp} = -2\tan^{-1}\left(\frac{\sqrt{\sin^2\theta_i - n^2}}{\cos\theta_i}\right) \approx \underline{\underline{-123,0^\circ}}$$

d) In $\delta = 1/\alpha$ the electric field drops to $1/e$.

$$\alpha = \frac{2\pi n_t}{\lambda_0}\left[\left(\frac{n_i}{n_t}\right)^2 \sin^2\theta_i - 1\right]^{1/2}$$

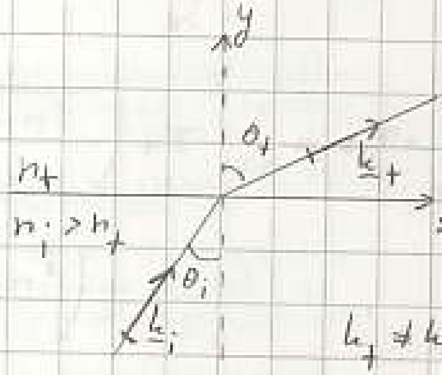
At $\theta_i = 77^\circ$ $\qquad \underline{\underline{\delta \approx 112\,nm}}$

At $\theta_i = 87^\circ$ $\qquad \underline{\underline{\delta \approx 99\,nm}}$

The penetration depth decreases as the angle of incidence comes closer to 90°.

4) When the angle of incidence exceeds critical angle the wave vector gets imaginary for the transmitted wave.

$$n_i \sin\theta_i = n_t \sin\theta_t$$
$$\theta_t = \sin^{-1}\left(\frac{n_i}{n_t}\sin\theta_i\right)$$



$k_t \neq k_i$

$$\frac{k_i}{n_i} = \frac{k_t}{n_t}$$
$$k_t = \frac{n_t}{n_i} k_i$$

Transmitted wave vector:

$$\underline{k}_t = k_{ty}\hat{y} + k_{tx}\hat{x},$$

where $k_{ty} = k_t \cos\theta_t$

and $k_{tx} = k_t \sin\theta_t$.

At incidence angles greater than the critical angle the angle of refraction gets into form of

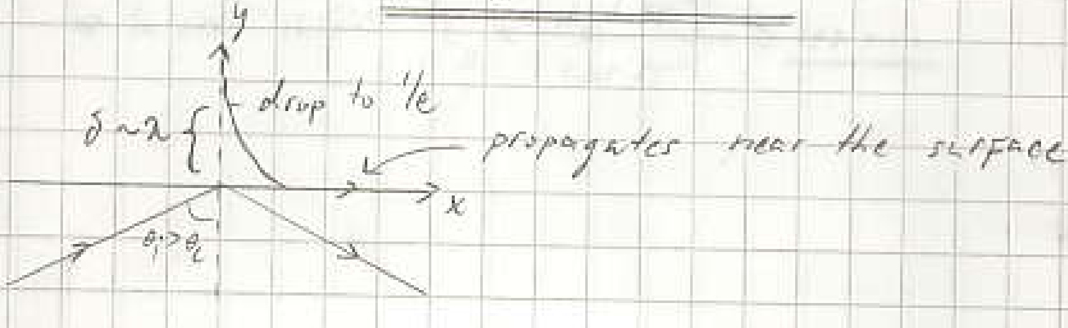
$$\theta_t(\theta_i > \theta_c) = 90^\circ + i\gamma.$$

Now we get: $k_{ty} = k_t \cos[\theta_t(\theta_i > \theta_c)] = \underbrace{i\alpha}_{\text{Fully imaginary}}$

$$k_{tx} = k_t \sin[\theta_t(\theta_i > \theta_c)] = \underbrace{k_t\left(\frac{n_i}{n_t}\sin\theta_i\right)}_{\text{Fully real}}$$

The transmitted wave takes now a form of

$$E_t \propto e^{-i(\omega t - \underline{k}\cdot\underline{r})} = e^{-i[\omega t - (i\alpha y + k_{tx}x)]}$$
$$= \underline{\underline{e^{-\alpha y}e^{-i(\omega t + k_{tx}x)}}}$$



Photonics 17 Risto Montonen

5) Gaus-Hänchen shift

Stationary phase method solves the Gaus-Hänchen shift as

$$d = -\frac{\lambda_0}{2\pi n_i}\,\frac{\partial\phi(\theta_i)}{\partial\theta_i}.$$

Phase shifts:

$$\phi_{\parallel}(\theta_i) = -2\tan^{-1}\left(\frac{\sqrt{\sin^2\theta_i - n^2}}{n^2\cos\theta_i}\right) + \pi, \quad n = \frac{n_t}{n_i}$$

$$\phi_{\perp}(\theta_i) = -2\tan^{-1}\left(\frac{\sqrt{\sin^2\theta_i - n^2}}{\cos\theta_i}\right)$$

Gaus-Hänchen shift is calculated by solving the derivative. Let's calculate the derivative in Wolfram Alpha.

$$\Rightarrow d_{\perp}(\theta_i) = \frac{\lambda_0}{\pi n_i}\,\frac{\sin\theta_i}{\sqrt{\sin^2\theta_i - n^2}}$$

$$d_{\parallel}(\theta_i) = -\frac{\lambda_0}{\pi n_i}\,\frac{n^2(n^2-1)\tan\theta_i\sec\theta_i}{\sqrt{\sin^2(x) - n^2}\,(n^4 - n^2\sec^2\theta_i + \tan^2\theta_i)}$$

At $\lambda_0 = 850\,nm$, $n_i = 1{,}460$, $n_t = 1{,}470$, and $\theta_i = 85°$ Gaus-Hänchen shift becomes to

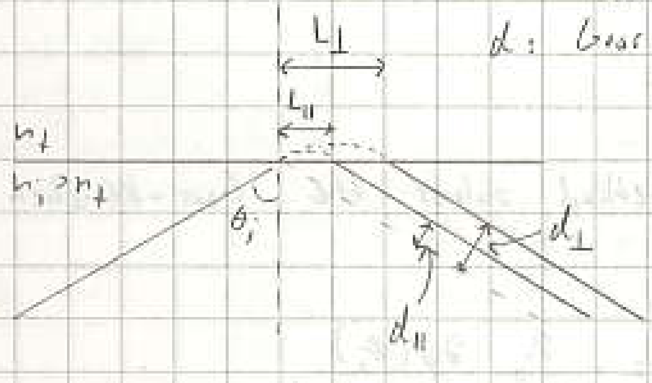
$$d_{\perp}(85°) \approx \underline{17{,}7\,nm}$$

$$\Rightarrow \text{lateral shift} \quad L_{\perp} = \frac{d_{\perp}(85°)}{\cos 85°} \approx \underline{\underline{203\,nm}}$$

$$d_{\parallel}(85°) \approx \underline{17{,}3\,nm}$$

$$\Rightarrow \text{lateral shift} \quad L_{\parallel} = \frac{d_{\parallel}(85°)}{\cos 85°} \approx \underline{\underline{198\,nm}}$$

L: lateral Goos-Hänchen shift
d: Goos-Hänchen shift



There is difference between $d_\parallel$ and $d_\perp$ which makes it possible to separate p and s polarizations. Furthermore, there is different phase change for p and s reflections. Although the shift is small it leaves it possible to engineer metamaterials for which the shift is bigger and more applicable in practice.