Photonics 17 Risto Montonen

1) a) Maxwell's equations in terms of total charge density:

(1) $\nabla \cdot \underline{E} = \frac{\rho}{\varepsilon_0}$, Gauss's law , ρ = charge density,
ε_0 = vacuum permittivity, $\underline{E}$ = electric field

(2) $\nabla \cdot \underline{B} = 0$, Gauss's law for magnetism, $\underline{B}$ = magnetic field

(3) $\nabla \times \underline{E} = -\frac{\partial B}{\partial t}$, Faraday's law for induction, t = time

(4) $\nabla \times \underline{B} = \mu_0 \left(\underline{J} + \varepsilon_0 \frac{\partial E}{\partial t} \right)$, Ampère's circuital law,
μ_0 = vacuum permeability,
$\underline{J}$ = current density

In vacuum there is no charge ($\rho = 0$) and no current ($\underline{J} = 0$).

Let's take curl of (3) and use rule $\nabla \times (\nabla \times \underline{A}) = \nabla(\nabla \cdot \underline{A}) - \nabla^2 \underline{A}$

$$\nabla \times (\nabla \times \underline{E}) = \nabla \times \left(-\frac{\partial B}{\partial t} \right)$$

$$\underbrace{\cancel{\nabla(\nabla \cdot \underline{E})}^{\,0}}_{\text{no charge}} - \nabla^2 \underline{E} = -\frac{\partial}{\partial t} (\nabla \times \underline{B})$$

no current

$$\nabla^2 \underline{E} = \frac{\partial}{\partial t} \left[\mu_0 \left(\cancel{\underline{J}}^{\,0} + \varepsilon_0 \frac{\partial E}{\partial t} \right) \right]$$

$$\underline{\underline{\nabla^2 \underline{E} = \varepsilon_0 \mu_0 \frac{\partial^2 \underline{E}}{\partial t^2}}}$$ wave equation for electric field

Wave equation for magnetic field in similar way but we start with equation (4).

$$\Rightarrow \underline{\underline{\nabla^2 \underline{B} = \varepsilon_0 \mu_0 \frac{\partial^2 \underline{B}}{\partial t^2}}}$$

b) Harmonic plane wave propagates in linear dielectric with homogeneous ε and $\mu_r = 1$.

Electric field is linearly polarized in x-direction

$$\underline{E} = E_0 e^{-i(\omega t - kz)} \hat{x}$$

In general $\underline{E} = \underline{E}_0 e^{-i(\omega t - \underline{k}\cdot\underline{r})} = (E_{0x}\hat{x} + E_{0y}\hat{y} + E_{0z}\hat{z}) e^{-i(\omega t - \underline{k}\cdot\underline{r})}$

Solutions of form $e^{-i(\omega t - \underline{k}\cdot\underline{r})}$ are called plane wave solutions. If equations are assumed to have plane wave solutions spatial derivatives could be changed as

$$\nabla \to i\underline{k}$$
$$\nabla\cdot \to i\underline{k}\cdot$$
$$\nabla\times \to i\underline{k}\times \; .$$

Time derivati could be changed as $\frac{\partial}{\partial t} = -i\omega$.

Using these relations we get following Maxwell's equations in dielectric media with no free charge ($\rho = 0$) and no free current ($\underline{J} = 0$).

$$\underline{D} = \varepsilon \underline{E} = \varepsilon_r \varepsilon_0 \underline{E} \quad , \underline{J} = \sigma \underline{E} \quad , \underline{B} = \mu \underline{H} = \mu_r \mu_0 \underline{H} = \mu_0 \underline{H}$$

(1) $\nabla \cdot \underline{D} = \rho$ (3) $\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$

(2) $\nabla \cdot \underline{B} = 0$ (4) $\nabla \times \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t}$

$\Rightarrow$ (1) $\underline{k} \cdot \underline{D} = 0$ (3) $\underline{k} \times \underline{E} = \omega \underline{B}$

(2) $\underline{k} \cdot \underline{B} = 0$ (4) $\underline{k} \times \underline{B} = -\omega \mu_0 \varepsilon_0 \varepsilon_r \underline{E}$, $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$

$$= -\frac{\omega}{c^2} \varepsilon_r \underline{E}$$

Let's calculate B-field from eq. (3), $\underline{k} = k\hat{z}$

$$\underline{k} \times \underline{E} = \omega \underline{B} \Leftrightarrow \underline{B} = \frac{1}{\omega}\underline{k} \times \underline{E} = \frac{1}{\omega}(k\hat{z}) \times \left(E_0 e^{-i(\omega t - kz)}\hat{x}\right)$$

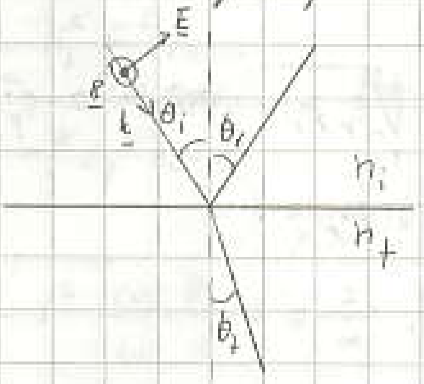
$$= \frac{1}{\omega}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & k \\ E_0 e^{-i(\omega t - kz)} & 0 & 0 \end{vmatrix}$$

$$k = \frac{\omega}{v} = \frac{\omega n}{c} \qquad = \frac{1}{\omega} k E_0 e^{-i(\omega t - kz)}\hat{y}$$

$$= \underline{\underline{\frac{n}{c} E_0 e^{-i(\omega t - kz)}\hat{y}}}$$

B-field has the same phase than E-field, amplitude is scaled by factor n/c, and polarization is perpendicular to E-field polarization.

2) a) TM (transverse magnetic) wave: $\underline{E}$ in the plane of incidence, $\parallel$ or p.



Fresnell equations:

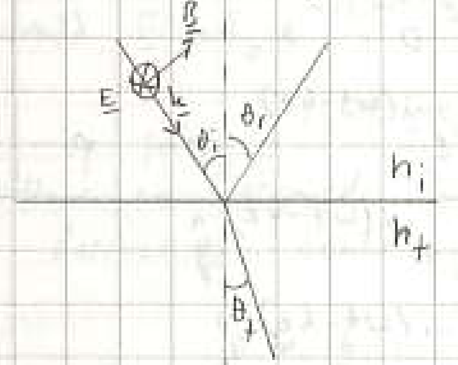
$$r_{\parallel} = \left(\frac{E_{0r}}{E_{0i}}\right)_{\parallel} = \frac{n_t \cos\theta_i - n_i \cos\theta_t}{n_i \cos\theta_t + n_t \cos\theta_i}$$

$$t_{\parallel} = \left(\frac{E_{0t}}{E_{0i}}\right)_{\parallel} = \frac{2n_i \cos\theta_i}{n_i \cos\theta_t + n_t \cos\theta_i}$$

Normal incidence $\Rightarrow \theta_i = \theta_t = \theta = 0 \Rightarrow \cos\theta = 0$

$$\Rightarrow r_{\parallel} = \frac{n_t - n_i}{n_i + n_t} \quad \text{and} \quad t_{\parallel} = \frac{2n_i}{n_i + n_t}$$

TE (transverse electric) wave: $\underline{B}$ in the plane of incidence, $\perp$ or s.



$$r_{\perp} = \left(\frac{E_{or}}{E_{oi}}\right)_{\perp} = \frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t} \overset{\text{Normal incidence}}{=} \frac{n_i - n_t}{n_i + n_t}$$

$$t_{\perp} = \left(\frac{E_{ot}}{E_{oi}}\right)_{\perp} = \frac{2n_i \cos\theta_i}{n_i \cos\theta_i + n_t \cos\theta_t} = \frac{2n_i}{n_i + n_t}$$

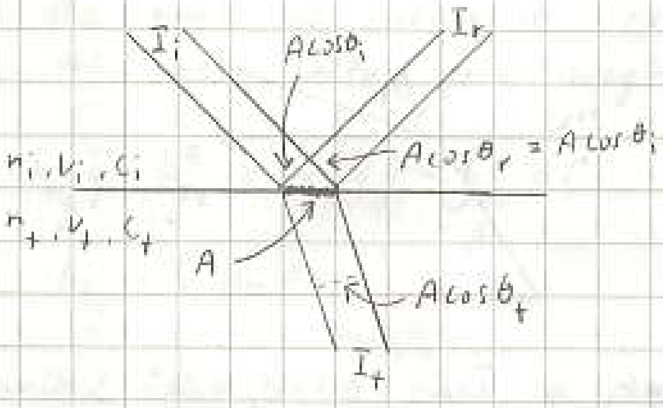
b) Irradiance (power per unit area)

$$I = \frac{nc\varepsilon_0}{2} E_0^2 \quad , \; n = \sqrt{\varepsilon_r \mu_r} \Leftrightarrow n^2 = \varepsilon_r \mu_r$$

$$= \frac{\cancel{n} c \varepsilon_r \varepsilon_0 \mu_r}{2n^{\cancel{2}}} E_0^2 \quad , \; \mu_r = 1$$

$$= \frac{c\varepsilon}{2n} E_0^2 \quad , \; \frac{c}{n} = v$$

$$= \frac{v\varepsilon}{2} E_0^2$$



Reflectance = $\dfrac{\text{reflected power}}{\text{incoming power}}$

normal incidence

$$R = \frac{I_r \cancel{A\cos\theta_i}}{I_i \cancel{A\cos\theta_i}} = \left(\frac{E_r}{E_i}\right)^2 = r^2 = \underline{\underline{\left(\frac{n_i - n_t}{n_i + n_t}\right)^2}}$$

Transmittance:

$$T = \frac{I_t \cancel{A}\cos\theta_t}{I_i \cancel{A}\cos\theta_i} = \frac{\frac{v_t \varepsilon_t}{\cancel{2}} E_t^2 \cos\theta_t}{\frac{v_i \varepsilon_i}{\cancel{2}} E_i^2 \cos\theta_i} = \frac{v_t \varepsilon_{rt} \cancel{\varepsilon_0} E_t^2 \cos\theta_t}{v_i \varepsilon_{ri} \cancel{\varepsilon_0} E_i^2 \cos\theta_i}$$

$\varepsilon_r = n^2$ at $\mu_r = 1$ and $v = c/n$

$$= \frac{\frac{c}{\cancel{n_t}} n_t^{\cancel{2}} E_t^2 \cos\theta_t}{\frac{c}{\cancel{n_i}} n_i^{\cancel{2}} E_i^2 \cos\theta_i} = \frac{n_t}{n_i}\left(\frac{E_t}{E_i}\right)^2 \frac{\cos\theta_t}{\cos\theta_i}$$

normal incidence

$$= \frac{n_t}{n_i} t^2 \frac{\cos\theta_t}{\cos\theta_i} = \frac{n_t}{n_i}\left(\frac{2n_i}{n_i + n_t}\right)^2$$

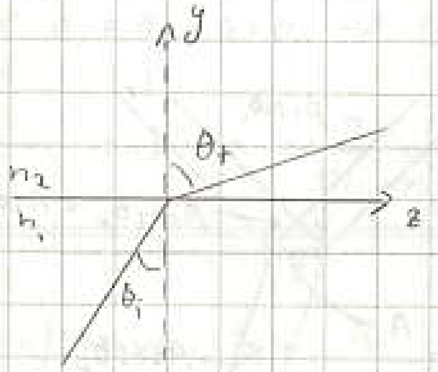
$$= \frac{n_t}{\cancel{n_i}} \frac{4n_i^{\cancel{2}}}{(n_i + n_t)^2} = \underline{\underline{\frac{4n_i n_t}{(n_i + n_t)^2}}} \neq t^2$$

N.B.

c) $n_i = 1$, $n_t = 1{,}5$

$$\Rightarrow R = \left(\frac{1 - 1{,}5}{1 + 1{,}5}\right)^2 = 0{,}04 = \underline{\underline{4\%}}$$

3) a)



Critical angle of TIR at $\theta_t = 90°$
Using Snell's law we get

$$n_1 \sin\theta_i = n_2 \sin\theta_t \quad , \sin\theta_t = 1 \text{ at } \theta_t = 90°$$

$$\theta_i = \sin^{-1}\frac{n_2}{n_1} = \theta_c$$

When $n_1 = 1,45$ and $n_2 = 1,33$ the critical angle is $\underline{\theta_c \approx 66,5°}$.

b) When θ_i is greater than θ_c phase shift occurs.

Phase shift for s-polarization

$$\tan\left(\frac{\phi_\perp}{2}\right) = \frac{-\sqrt{\sin^2\theta_i - n^2}}{\cos\theta_i} \quad , \quad n = \frac{n_2}{n_1}$$

at $\theta_i = 70° \Rightarrow \phi_\perp \approx \underline{-61,7°}$

Phase shift for p-polarization

$$\tan\left(\frac{\phi_\parallel - \pi}{2}\right) = \frac{-\sqrt{\sin^2\theta_i - n^2}}{n^2\cos\theta_i}$$

at $\theta_i = 70° \Rightarrow \phi_\parallel \approx \underline{109,3°}$

Goos-Hänschen shift is a consequence of the phase shift. Linearly polarized light is changed to elliptically polarized after TIR.

c) When the angle of incidence exceeds critical angle the wave vector gets imaginary

$$\underline{k} = k_{ty}\,\hat{y} + k_{tz}\,\hat{z} = i\alpha\,\hat{y} + k_{tz}\,\hat{z}$$

where $k_{tz} = k_1 \sin\theta_i$.

Transmitted wave takes now a form

$$E_t \propto e^{-i(\omega t - \underline{k}\cdot \underline{r})} = e^{-i(\omega t - (i\alpha y + k_{tz} z))} = e^{-\alpha y}\, e^{-i(\omega t - k_{tz} z)}$$

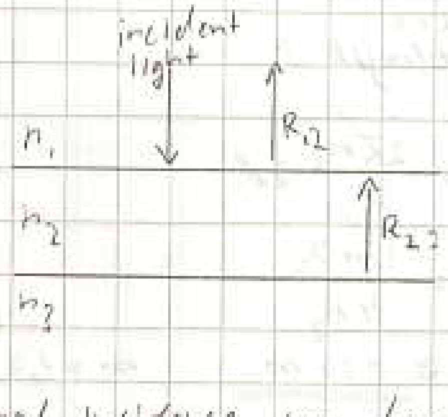
Here α is an attenuation coefficient for the electric field penetrating to medium 2.

$$\alpha = \frac{2\pi n_2}{\lambda_0}\left[\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_i - 1\right]^{1/2}$$

free space "vacuum" wavelength

In $\delta = 1/\alpha$ the electric field drops to $1/e$. δ is called penetration depth and equals $\underline{0{,}57\ \mu m}$ when $\lambda_0 = 1{,}064\ \mu m$, $n_1 = 1{,}45$, $n_2 = 1{,}33$, and $\theta_i = 70°$.

4) a)



For normal incidence we have

$$R_{12} = \left(\frac{n_2 - n_1}{n_1 + n_2}\right)^2 \quad \text{and} \quad R_{23} = \left(\frac{n_3 - n_2}{n_2 + n_3}\right)^2$$

$$R_{12} = R_{23}$$

$$\left(\frac{n_2-n_1}{n_1+n_2}\right)^2 = \left(\frac{n_3-n_2}{n_3+n_2}\right)^2$$

$$\frac{n_2-n_1}{n_1+n_2} \overset{+}{(-)} \left(\frac{n_3-n_2}{n_3+n_2}\right)$$

$$(n_2-n_1)(n_3+n_2) = (n_1+n_2)(n_3-n_2)$$

$$\cancel{n_2 n_3}+n_2^2-n_1n_3-\cancel{n_1n_2} = n_1n_3-\cancel{n_1n_2}+\cancel{n_2n_3}-n_2^2$$

$$\cancel{2}n_2^2 = \cancel{2}n_1n_3$$

$$\underline{\underline{n_2 = \overset{+}{(-)}\sqrt{n_1n_3}}}$$

b) $n_{air} = n_1 = 1$, $n_3 = n_{BK7}(\lambda = 1{,}064\,\mu m) \approx 1{,}507$

$\Rightarrow n_2 \approx \underline{\underline{1{,}228}}$.

For a antireflection coating we need to have a thickness that generates $m\pi$ phase difference between the two reflections, m is positive integer.

Phase delay in the coating equals $k2d$, where

$$k = \frac{2\pi}{\lambda} = \frac{2\pi n_2}{\lambda_0}$$

medium wavelength

vacuum wavelength

$$\Rightarrow \quad m\cancel{\pi} = \frac{2\cancel{\pi} n_2}{\lambda_0}2d$$

$$d = \frac{m\lambda_0}{4n_2}$$

$$\approx \underline{\underline{0{,}22\,m}} \quad , m = 1,2,3,\ldots$$

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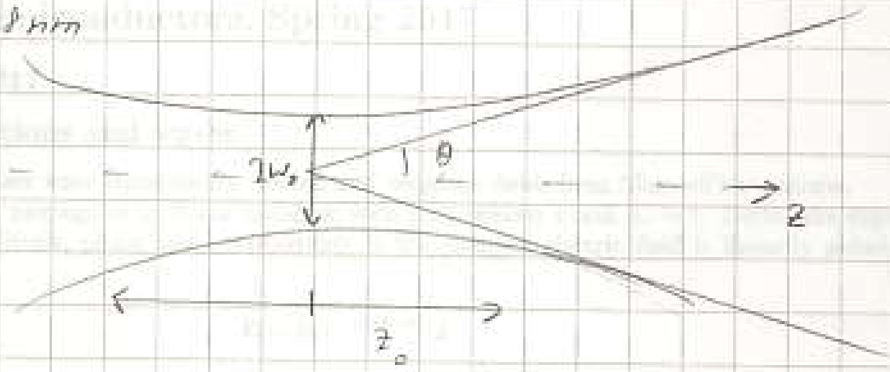
5) $\lambda = 632{,}8\,nm$



$2w_0 = 1\,mm$

$w_0 = 0{,}5\,mm$

At z_0 from center the on-axis intensity is $1/2$ of the peak intensity.

Divergence: $\theta = \dfrac{2\lambda}{\pi(2w_0)} \approx 0{,}4\,mrad = \underline{0{,}023^\circ}$

Rayleigh range: $z_0 = \dfrac{\pi w_0^2}{\lambda} \approx \underline{1{,}24\,m}$

Beam diameter at 10 m: for large z

$$2w = 2w_0\left[1+\left(\frac{z}{z_0}\right)^2\right]^{1/2} \approx 2w_0\left(\frac{z}{z_0}\right) \approx \underline{8{,}1\,mm}$$